Signals and Systems

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Background and Acknowledgements

This material has been developed for the first course in Signal and Systems. The content is derived from the author's educational, technical and management experiences, in-addition to teaching experience. Many other sources, including the following specific sources, have also informed the content and format of this text:

- Nilsson, J. Electrical Circuits. (2004) Pearson.
- Oppenheim, A. Signals & Systems (1997) Prentice Hall
- Stremler, F. Introduction to Communication Systems (1990) Addison
- Lathi, B. Modern Digital and Analog Communication Systems (1998) Oxford University Press
- MathWorks. MATLAB Reference Material Version R2000a. (2007) MathWorks

I would like to give special thanks to my students and colleagues for their valued contributions in making this material a more effective learning tool.

I invite the reader to forward any corrections, additional topics, examples and problems to me for future revisions.

Thanks,
Izad Khormaee
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1.1. Introductions

Study of signals and systems leverages mathematics, computer solutions, understanding of science and system engineering in order to analyze system behavior, design systems and derive information from signals. Signal and systems application can be found in a broad range of fields including:

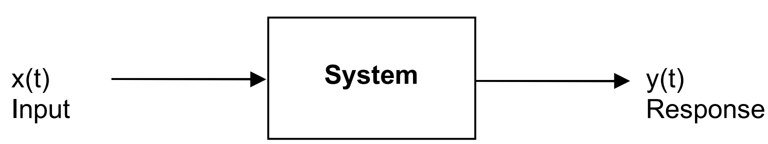
- Communication
- Aeronautics and Astronautics
- Circuit design
- Acoustics and visuals
- Seismology and Geology
- Biomedical Engineering
- Energy generation
- Distribution systems
- Chemical Process Control
- Speech Processing
- Financial Analysis and Forecasting

Although the underlying phenomenon or effect being studied in each field may be dramatically different, they all share two basic features:

- Signals
  Signals are defined as functions that are dependent on one or more independent variables and carry information about the behavior or nature of one or more phenomenon.

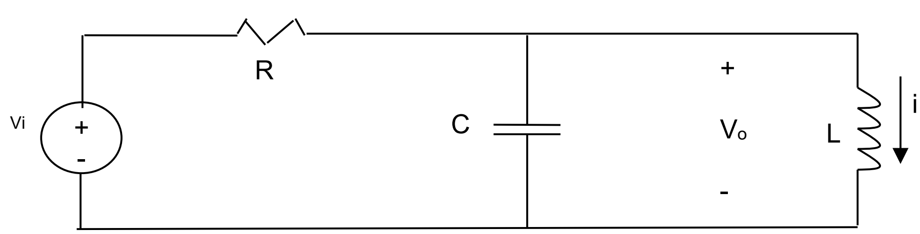
  In this text we will focus on signals that depend only on a single independent variable. For example x(t) or x[n].

- Systems
  Systems are defined to respond to a particular signals by producing another signals which has a set of desired characteristics



Here are some examples of signals and systems applications:

- Electrical Circuits
  Voltage value over time may be considered a signal → x(t)
  Voltage or current in any other part of circuit may be used as system response → $y_1(t)$ and $y_2(t)$



- Seismology
  Signal may be the signal generated from the impact of some physical device with ground → x(t)
  Response may be the reflection of signal as it bounces off different layers → y(t)
- Financial Market
  Economic parameters such as interest rate, earning, past price → $x_1(t)$, $x_2(t)$, $x_2(t)$,…

Output may be the future price of stocks → y(t).

- Most commonly signal and systems techniques are used to:

  - Analyze and characterize existing systems
  - Design systems to process signals based on a set of rules
  - Enhance and restore signals
  - Controlling characteristic of given systems based on input signals, system behavior and other systems.

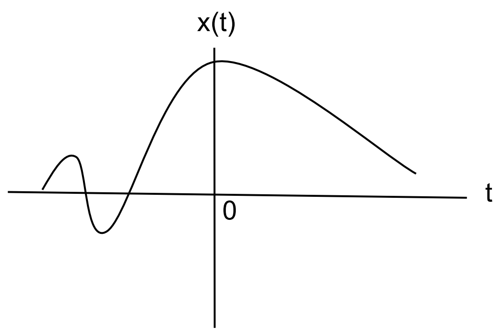
The remainder of this text provides a broad coverage of signal and systems with a focus on linear systems that are time invariant.

1.2. Continuous-Time (CT) and Discrete-Time(DT) Signals

As discussed earlier, the focus here is on signals with single independent variable. Further the default independent variable will be time, t. This enables the most efficient coverage of the topics but it is important to remember that the concepts may be applied to other signals with other types of independent variables.

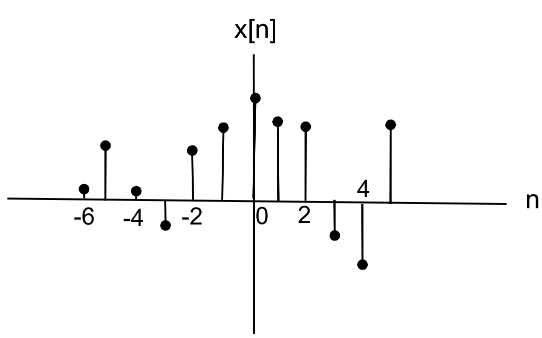
Natural phenomenon signals are continuous which means at any point time there is a value associated with the signal. This type of signal is referred to as Continuous-Time signal where the independent variable is continuous. For example, function $x(t)=10\sin(20\pi t)$ represent a continuous function.

In continuous-time, independent variable is represented by t (real number). In general continuous time signals are plotted with connected lines as shown below:

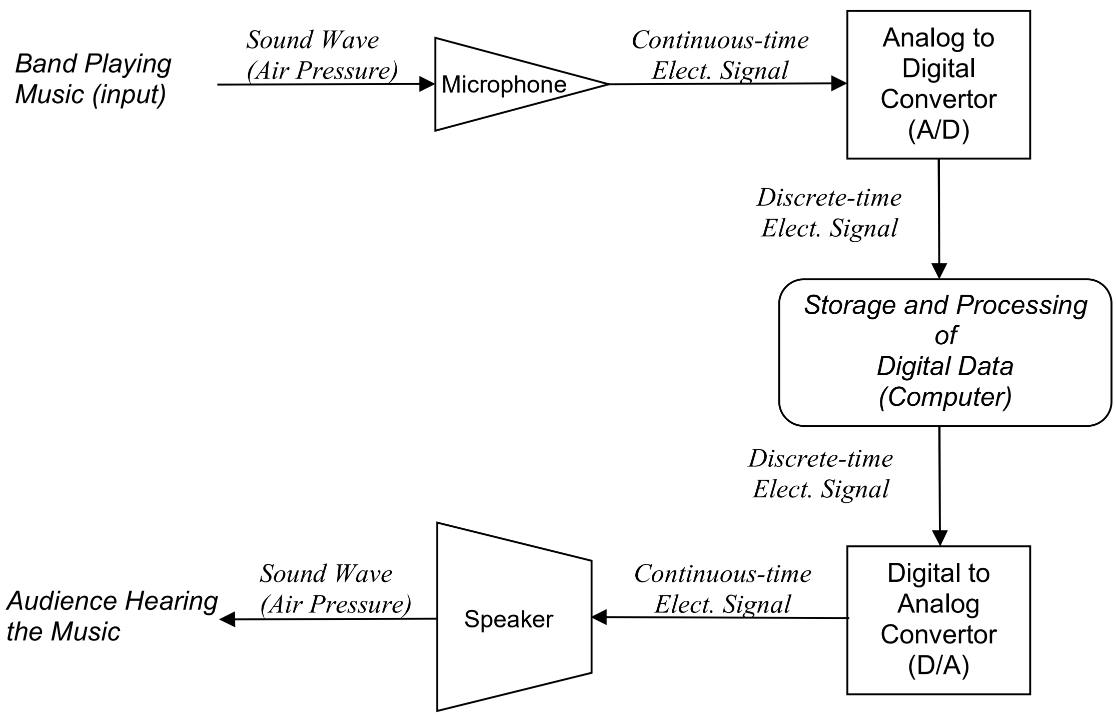


The second type of signal used in Signal & Systems is the Discrete-Time signal where the independent variable only takes discrete values. Discrete-Time signal allows for signal to be constructed out of discrete observed values. Each discrete value is a sample and we are not able of make any definite statements about the signal value between the sample points. In many cases, we make assumptions based on the underlying system characteristic in order to approximate the values between the sample points.

x[n] is used to represent Discrete-Time signal (Note the use of "[" instead of "(" and "n" instead of "t"). n is an integer number. For example, function $x[n] = 10 \cos(10\pi n)$ represent a discrete function. In general, Discrete-Time functions are plotted as stems:



Although we sense and effect our environment in continuous-time, efficiency of digital (computer) systems has encouraged the use of Discrete-Time to approximate and model systems. Digital systems are less expensive and more flexible in storing and processing system data. These facts have resulted in majority of systems designed to convert Continuous-Time information to Discrete-Time for storage and processing and then convert them back to Continuous-Time for usage. Below is an example of such a process in common audio systems:

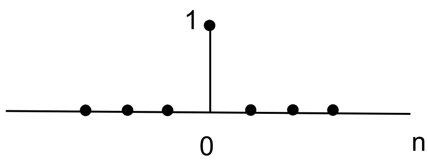


1.3. Unit Impulse and Unit Step Functions

Unit Impulse and Unit Step functions are two ideal signals specifically defined as a tool for signal processing. Both signals are crucial in our ability to model systems mathematically. In this section we will define these functions.

❖ Discrete-Time Unit Impulse and Unit Step Function

- Unit Impulse, $\partial[n]$ (or unit sample)
  Unit Impulse is equal to 1 only when the independent variable is equal to 0; otherwise Unit Impulse is 0.



$$\delta\,[n] \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

δ *"Greek character Delta"*

Here are a couple of relationships that explain the reason for also referring to impulse function as sample function.

$$\sum_{n=-\infty}^{\infty} \mathrm{x}[\mathrm{n}]\delta[\mathrm{n}] = \mathrm{x}[0]\delta[\mathrm{n}] = \mathrm{x}\,[0]$$

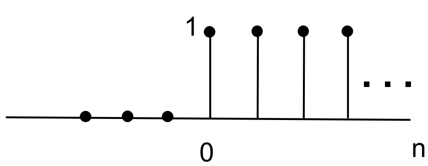
To prove refer to the definition of δ[n] which says δ[n] is only 1 at n=0

The more general form of the above equation is shown below:

$$\sum_{n=-\infty}^{\infty} \mathrm{x}[\mathrm{n}]\delta[\mathrm{n} - \mathrm{n}_0] = \mathrm{x}[\mathrm{n}_0]\delta[\mathrm{n} - \mathrm{n}_0] = \mathrm{x}\,[\mathrm{n}_0]$$

Unit Impulse is commonly referred to as impulse function.

- Unit Step function, u[n]
  Step response is equal to 1 as long as the independent variable is larger or equal to 0.



$$\mathrm{u}[\mathrm{n}] \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Unit Step function is commonly referred to as simply Step function.

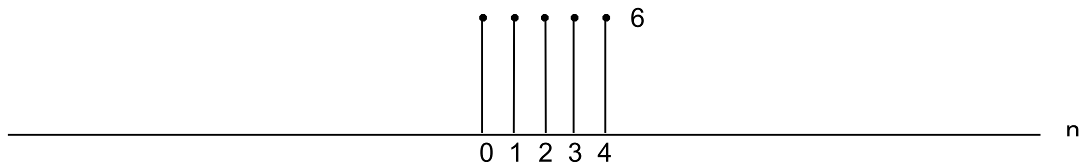
- Relationships between Step and Impulse functions

  * From Step to Impulse function conversion
  . δ[n]=u[n]-u[n-1]

  * From Impulse to Step Function conversion

u[n]= $\sum_{k=0}^{\infty} \delta[n-k]$

- Example - Write the following function in-terms of step functions and then again in terms of Impulse function.

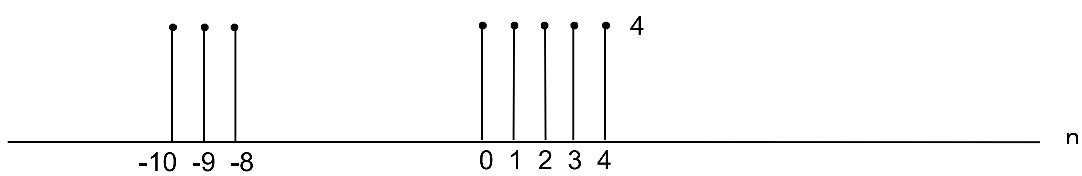


Solution:

Step Function Representation:
x[n] = 6u[t] - 6u[t-5]

Impulse Function Representation:
x[n]= 6{ δ[0] + δ[1] + δ[2] + δ[3] + δ[4] }

- Example - Write the following function in-terms of step functions.

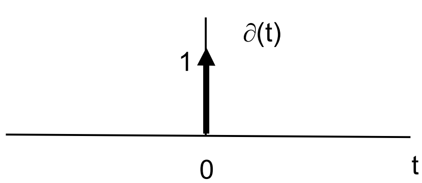


Solution:

<Student Exercise>

- Continuous-Time Unit Impulse and Unit Step Response

  - Unit Impulse, ∂(t) (or unit sample)
    Unit Impulse function has no duration but the unit area is 1.



$$\partial(t) \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

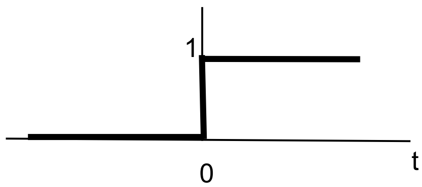
Unit Impulse is commonly referred to simply as impulse function. $\int_{-\infty}^{\infty} x(\tau)\delta(\tau)d\tau = x(0)$

This is true based on definition of $\delta(t)$ which say it is only 1 at t=0

Here is a more general form:

$$\int_{-\infty}^{\infty} \mathrm{x}(\tau)\delta(\tau - t_0)d\tau = \mathrm{x}(t_0)\delta(t - t_0) = \mathrm{x}(\mathrm{t}_0)$$

- Unit Step, u(t) function
  Step function is equal to 1 when t>0 and is equal otherwise. Step function is undefined at t=0.



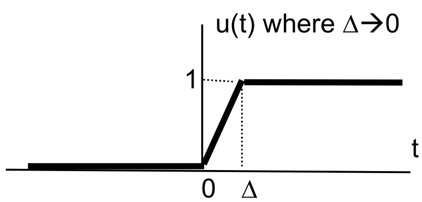
$$\mathrm{u(t)} \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

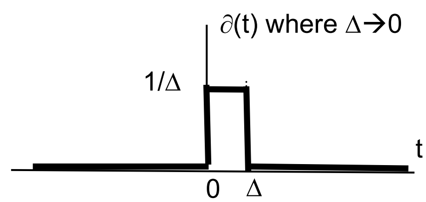
Unit Step function is commonly referred to simply as Impulse function

- Relationship between Impulse and Unit Functions
  Impulse function may be written in-terms of Unit Step Function using the following relationship:

$$\delta(t) = \frac{du(t)}{dt}$$

Below is the graphical representation:



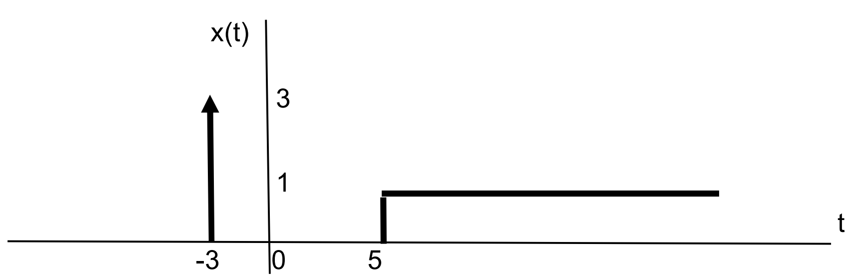


Another approach is to write Unit Step functions in-terms of Impulse Functions using:

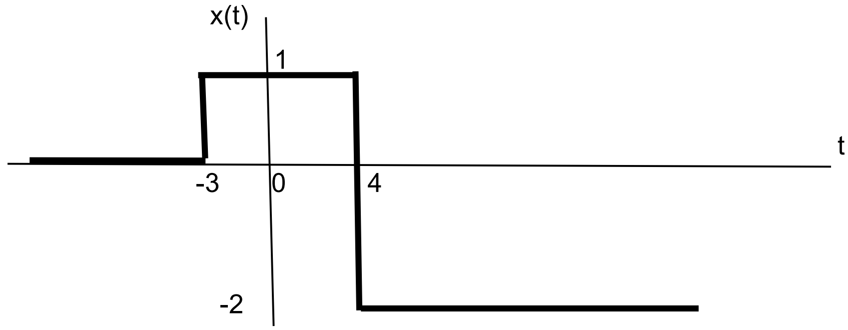
$$u(t) = \int_{0}^{\infty} \delta(\tau)d\tau$$

- Example – Draw the function x(t)=u(t-5) + 3∂(t+3).

Solution:



- Example – Write the function for the following graphs in-terms of Step functions.



Solution:

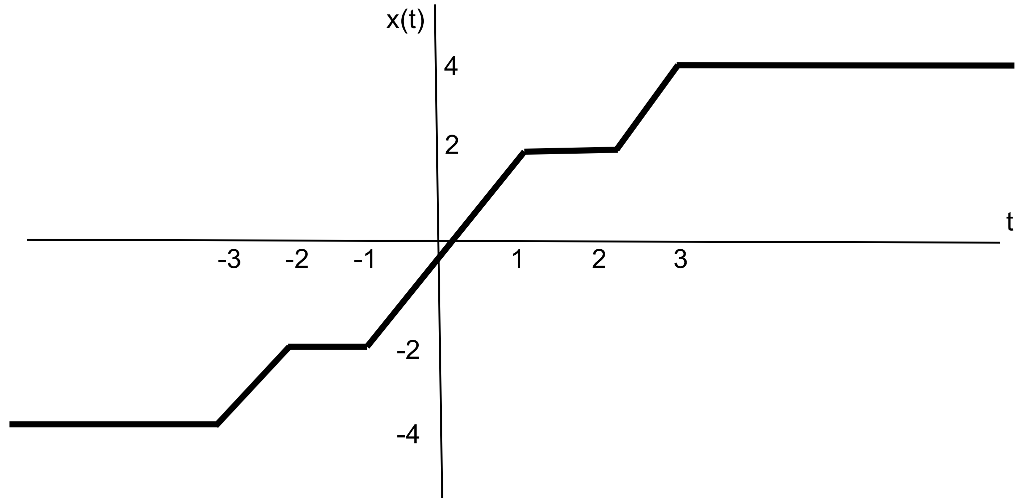
$x(t) = u(t+3) – 3u(t-4)$

- Example – Draw the function represented by:

$$x(t)= u(t + 3)u(-t + 3) + 5\partial(t - 4)$$

Solution:

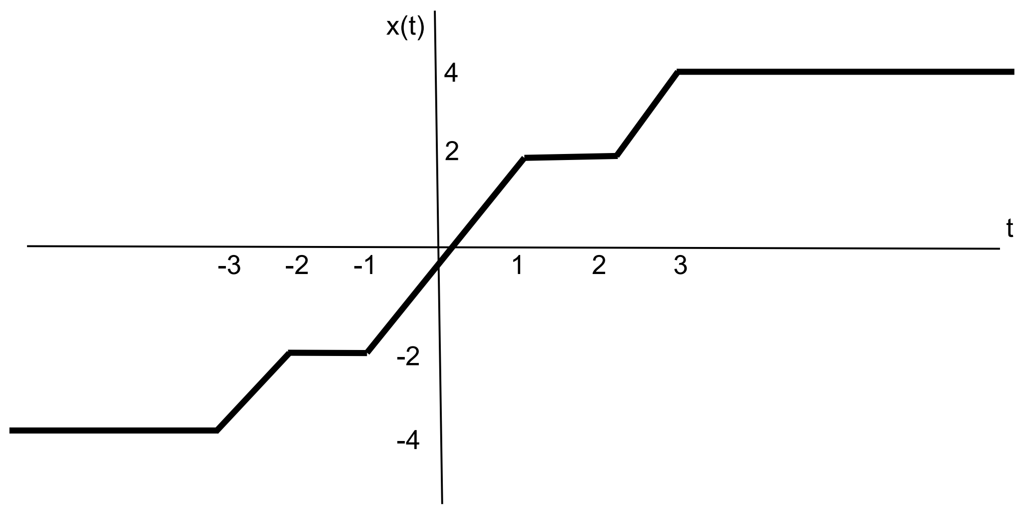
➢ Example – Determine the non-zero values of the function {x(t-6)u(t-9)} give the following x(t):



Solution:

The function is non-zero, {x(t-6)u(t-9)}=4, when t≥9.

➢ Example – Determine the non-zero values of the function {x(t-4)u(-t+1)} give the following x(t):



Solution:

Student Exercise

1.4. Signal Energy and Power

In signals and systems, the first step is to relate the signals to the physical quantities. In electrical engineering, the focus is on power and by extension energy since power defines the ability of any electrical systems to effect change or sense change.

In this section we will define three types of signals based on their energy profile. Note that for each concept introduced, it will be discussed with respect to Continuous-Time signal and with respect to Discrete-Time signal. In most cases, the two treatments will be similar but there are instances where the continuous (t) vs. discrete [n], effects the outcome.

Let's start from the basic concepts of Power (P) and Energy (E) in electrical engineering:

Power → $p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$

The total energy over the time interval $t_1 \le t \le t_2$

$$E = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

The average power over the time interval $t_1 \le t \le t_2$ is represented by

$$P = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

As discussed earlier, in the study of signal and systems, x(t) may be used to represent the magnitude of voltage therefore the energy equation for the Discrete-Time and Continuous-Time may be written as shown below:

- Continuous-Time Signal x(t) where |x(t)| is the amplitude of Complex value x(t)
  - The total energy over the time interval $t_1 \le t \le t_2$ → $E = \int_{t_1}^{t_2} |\, x(t)\,|^2 \; dt$
  - The average power over the time interval $t_1 \le t \le t_2$ → $P = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |\, x(t)\,|^2 \; dt$
- Discrete-Time Signal x[n] where |x[n]| is the amplitude of Complex value x[n]
  - The total energy over the time interval $n_1 \le n \le n_2$ → $E = \sum_{n=n_1}^{n_2} |\, x[n]\,|^2$
  - The average power over the time interval $n_1 \le n \le n_2$ → $P = \frac{1}{n_2 - n_1 + 1}\sum_{n=n_1}^{n_2} |\, x[n]\,|^2$

Next, the above derivations may be extended by allowing the independent variables to approach ∞:

- Continuous-Time signal - Energy and Average Power in interval $-\infty \leq t \leq \infty$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt \text{ and } P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

- Discrete-Time signal – Energy and Average Power in interval $-\infty \leq n \leq \infty$

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 \text{ and } P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

The E_∞ and P_∞. are used in classifying signals. The classification applies to both Discrete-Time signals and Continuous-Time signals. The three classes with respect to E_∞ and P_∞.are:

- $E_\infty < \infty$ → P_∞. = 0
  Signals that have finite total energy $E_\infty < \infty$ which in-turn will have Zero average power (P_∞.).
  For example: Signal that takes on value of 2 for 0 ≤ t ≤ 2 and zero otherwise. In this case E_∞=4 and P_∞=0.

- P_∞>0 → E_∞=∞
  Signal which have P_∞>0 which in-turn will have E_∞=∞.
  For example a constant signal x[n]=12 has infinite energy, but average power is 144.

- $P_\infty = \infty$ & $E_\infty = \infty$
  Signals that neither P_∞ nor $E_\infty$are finite.
  For example x(t)=t with both its average power and total energy are infinity.

1.5. Independent Variable Transformations

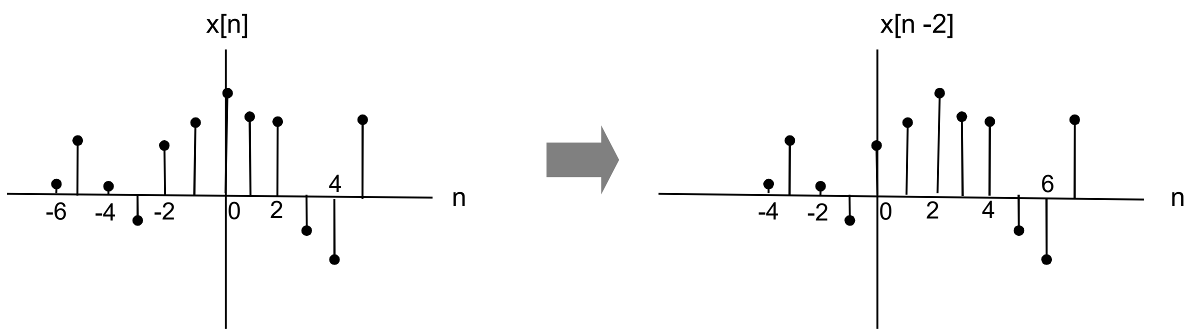
It is common in signal and systems to transform a system's response by transforming the independent variable (t or n). The three most common transformations used by engineers are time shift, time reversal and time scale. The remainder of this section will outline each of the three transformations and how they may be combined for a more complex transformation. As done earlier, each transformation is outlined for both Discrete-Time [n] and Continuous-Time {t}:

- Time Shift
  Time Shift delays or advances the signal by adjusting the independent variable.

  - $x(t) \rightarrow x(t - t_0)$
    - If $t_0 > 0$ the signal is delayed
    - If $t_0 < 0$ the signal is advanced

  - $x[n] \rightarrow x[n - n_0]$
    - If $n_0 > 0$ the signal is delayed
    - If $n_0 < 0$ the signal is advanced

    - Examples:

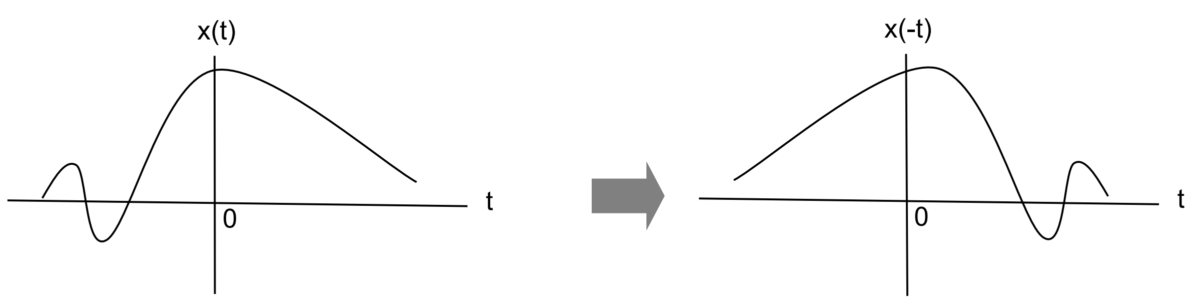


- Time Reversal
  Time Reversal will reflect the signal about the origin with respect with independent.

  - $x(t) \rightarrow x(-t)$

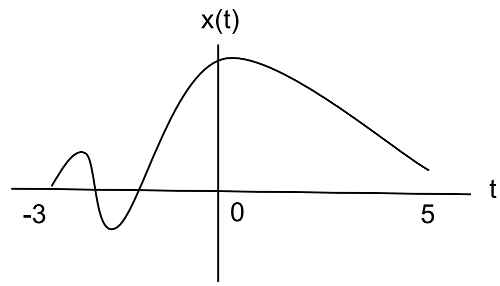
  - $x[n] \rightarrow x[-n]$

    - Examples:

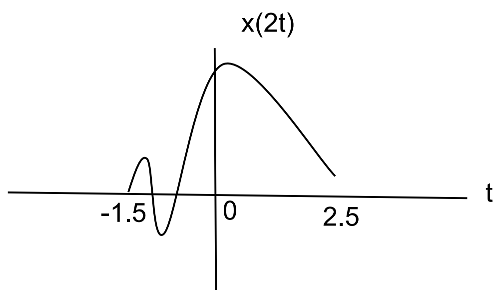


- Time Scaling
  Time Scaling expands or compresses by multiply independent variables with a constant.

  - $x(t) \rightarrow x(at)$
  - x[n] -> x[an]
  - Examples-Scaling
    Draw x(2t) for the following function, x(t):



    Solution:



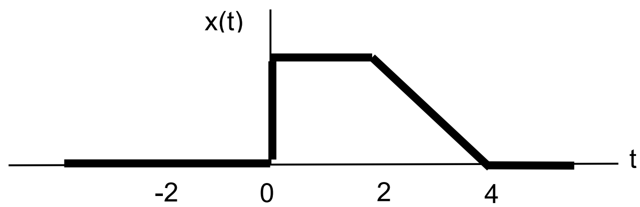
  - Example-Scaling
    Find the frequency of x(20t) when $x(t) = 25 \cos(1000\pi t)$.

    Solution:
        Student Exercise

- General Form
  The above three transformations may be combined into a single step in the general form shown here:

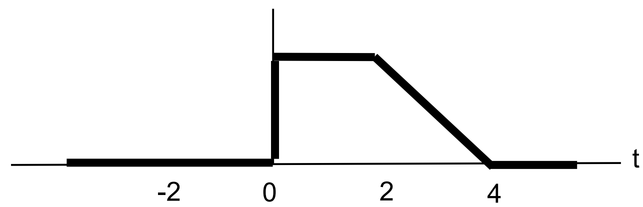
  - $x(t) \rightarrow x(at - b)$
  - x[n] -> x[an - b] where a & b are integers

❖ Example – Transformation
Transform x(t) to x(5t/3 + 2) for the x(t) shown below.

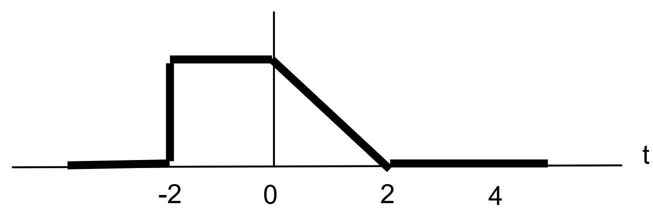


Solution:

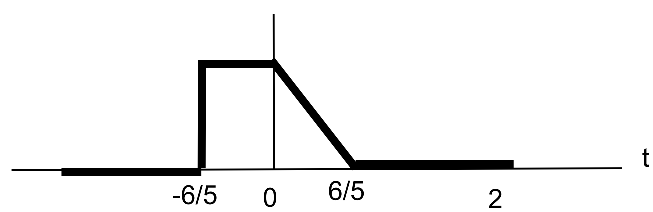
1. x(t) Original Signal



2. x(t+2) shifted signal

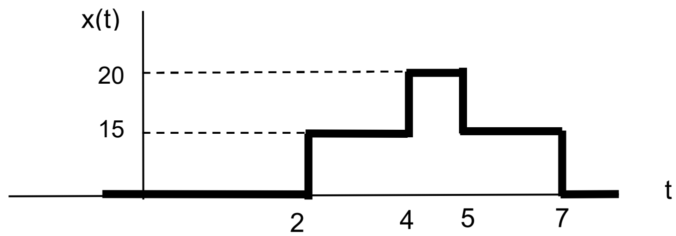


3. x(5t/3+2) scaled & shifted signal

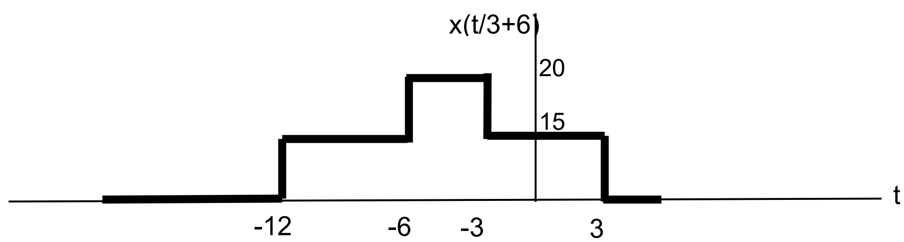


Note: This problem shifts first and scales second.

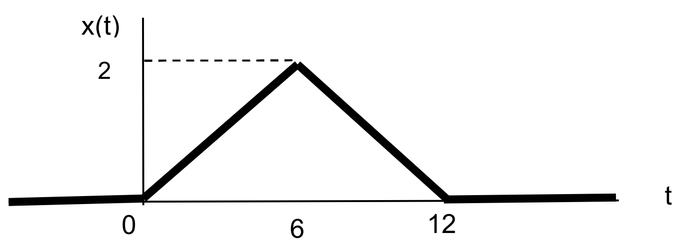
❖ Example – Transformation
Transform x(t) to x(t/3 + 6) for the x(t) shown below.



Solution: (shift first and then scale)



❖ Example – Transformation
Transform x(t) to x(3(t+4)) for the x(t) shown below.



Solution:
Student Exercise

1.6. Complex Exponential and Sinusoidal Signals

This section introduces the most important signal class, Complex Exponential and Sinusoidal Signals that is the foundation of signal and systems analysis. The first step is to re-examine the definition of periodic signal for Continuous-Time and Discrete-Time:

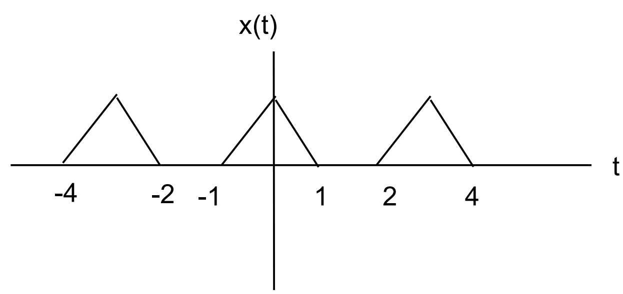
- Continuous-Time periodic signal with period T → x(t)=x(t+kT)
  - Smallest positive T (real number) that satisfies the above equation is called the Fundamental period T_0.

- Discrete-Time periodic signal with period N → x[n]=x(n+kN)
  - Smallest positive N (integer) that satisfies the above equation is called the Fundamental period N_0.

One property that is utilized to simplify signal and systems analysis is Symmetry about the independent variable origin. A signal may have even, odd or mixed Symmetry:

- Even Symmetry exists when:
  - x(t) =x(-t)
  - x[n]=x[-n]

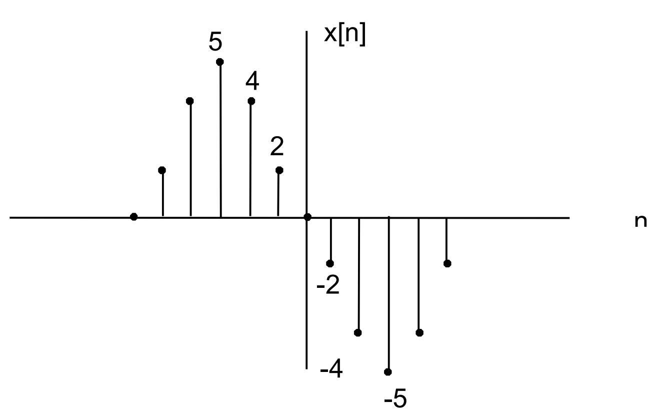
  For example, function x(t) has even symmetry.



- Odd Symmetry exists when:
  - x(t) =-x(-t)
  - x[n]=-x[-n]

    Note: odd symmetric function by definition must be 0 at n=0 or t=0.

    For example, function x[n] has odd symmetry.



- Examples – Odd/Even
  Are Sin() and Cos() functions odd or even?

  Solution:

  Student Exercise

If a signal is neither odd nor even then it must be a function with mixed symmetry. At times, it is useful to analyze the signal's odd and even components independently. Below is the process to find the even and odd components of any signal:

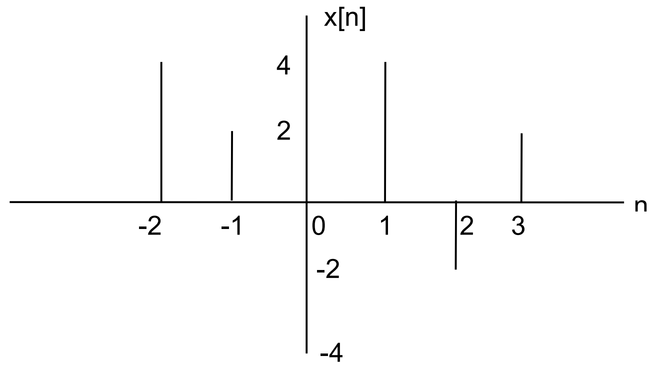
- Continuous-time
  - Even $\{x(t)\} = ½\{x(t) + x(-t)\}$
  - Odd $\{x(t)\} = ½\{x(t) – x(-t)\}$

- Discrete-time
  - Even $\{x[n]\} = ½\{x[n] + x[-n]\}$
  - Odd $\{x[n]\} = ½\{x[n] – x[-n]\}$

For example, $x(t) = 2t + 1$ is neither purely even nor purely odd but has mixed symmetry. Here is the process to find the even and odd parts of this mixed signal:

Even $\{x(t)\} = ½\{x(t) + x(-t)\} = ½ \{2t +1 -2t +1\} = 1$
Odd $\{x(t)\} = ½\{x(t) – x(-t)\} = ½ \{2t + 1 +2t -1\} = 2t$

- Examples – Odd/Even Part
  Find odd and even part of x[n]



Solution:
Student Exercise

- Examples – Odd/Even Part
  Find odd and even part of the following function:
  $x(t) = 3t + \cos(t)$

Solution:
Student Exercise

Now that we have the definition for periodic signals and symmetry, we are ready to introduce the Complex Exponential and Sinusoidal Signal classes. This class of signals is the basis of signal definition throughout this text and is the most common approach to signal definition in the industry. These signals serve as a basic building block of many common signals.

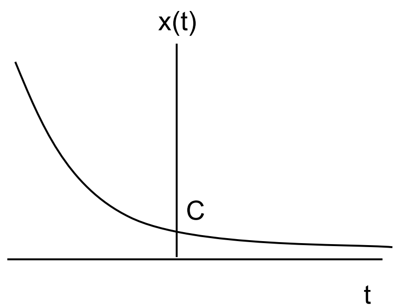
This section covers the Continuous-Time(CT) first, followed by Discrete-Time(DT) classes of Complex Exponential Sinusoidal Signals.

- Continuous-Time Complex Exponential Sinusoidal Signals
  The general form of Complex Exponential and Sinusoidal is best stated by the following:

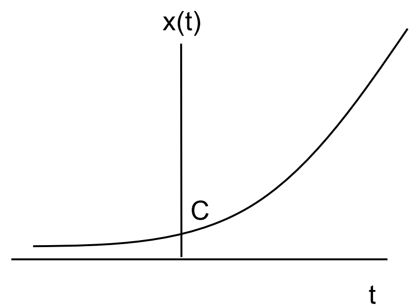
  $x(t) = Ce^{at}$ Where C and a are both complex numbers

  $$a = |a| e^{j\phi_a} = a_{real} + ja_{imag}$$
  $$C = |C| e^{j\phi_c} = C_{real} + jC_{imag}$$

  The simplest form of x(t) is when a=0 which resolves x(t) to a simple constant value. The next simplest form of x(t) occurs when both a and C are real. In this case x(t) resolves to a real exponential signal. Depending on the sign of a, x(t) may be growing and decaying exponential as shown below:



x(t) has the exponential decay form when (a<0)



x(t) has the exponential growth when (a>0)

  Another subclass of signals exists when "a" is pure imaginary which results in:

  $x(t) = e^{jw_0t}$ *Complex Periodic Exponential*.

  The Complex Periodic Exponential signals have a number of important properties which are listed below:

  - Periodicity
    Of course this signal is periodic which means:
    $x(t) = e^{jw_0t} = e^{jw_0(t+T)}$ Where
    $T_0 = \frac{2\pi}{w_0}$ is fundamental period and T is multiple of T_0

Here is the proof that the above equality is true:

$$x(t) = e^{jw_0(t+T)} = e^{jw_0t}e^{jw_0T}$$
$$apply\ Euler's\ \mathrm{Re}lation\ e^{jw_0T} = \cos w_0T + j\sin w_0T$$
$$Since\ T = n\frac{2\pi}{w_0}\ then\ e^{jw_0T} = 1 + j0 = 1$$
$$\therefore x(t) = e^{jw_0(t+T)} = e^{jw_0t}$$

- Sinusoidal Signal
  The Sinusoidal Signal $x(t) = A\cos(w_0t + \phi)$ is closely related to Complex Periodic Exponential (Real part of the signal) which is demonstrated below using Euler's Relation:

$$Note: Euler's\ \mathrm{Re}lation\ e^{\pm jb} = \cos b \pm j\sin b$$
$$x(t) = A\cos(w_0t + \varphi) = \frac{A}{2}e^{j\varphi}e^{jw_0t} + \frac{A}{2}e^{-j\varphi}e^{-jw_0t}$$
$$Another\ way\ to\ write\ it:$$
$$A\cos(w_0t + \varphi) = A\ \mathrm{Re}\,al\{e^{(jw_0t+\varphi)}\}$$
$$A\sin(w_0t + \varphi) = A\ \mathrm{Im}aginary\{e^{(jw_0t+\varphi)}\}$$

- Average Power and Total Energy
  Use the Average Power and Total Energy equations to calculate the corresponding values for Complex Periodic Exponential signal, $x(t) = e^{jw_0t}$.

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\ dt = \frac{1}{2T}\int_{-T}^{T}|e^{jw_0t}|^2\ dt = \frac{T+T}{2T} = 1$$

$$E_\infty = \int_{-\infty}^{\infty}|x(t)|^2\ dt = \int_{-\infty}^{\infty}|e^{jw_0t}|^2\ dt = \infty$$

Note: $|x(t)| = |e^{jw_0t}| = 1$; to prove this equality use Euler's Identity "$e^{ja} = \cos a + j\sin a$".

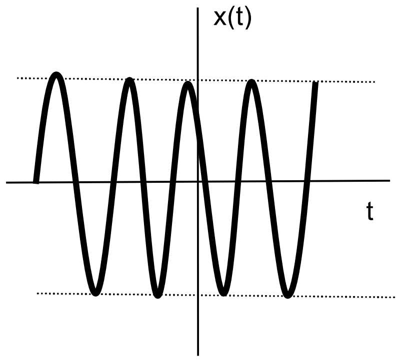
- Finally, the General form of Complex Period Exponential Signals $x(t) = Ce^{at}$ Where:

  $C = |C|e^{j\theta}$ C is complex represented in Polar Form
  $a = r + jw_0$ a is complex represented in Rectangular Form

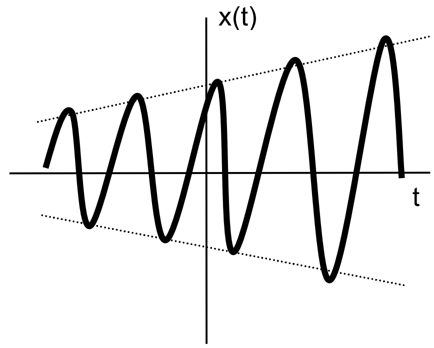
The above relationships may be used to rewrite $x(t) = Ce^{at}$ in a form that we are more familiar with, using Euler's Identity:

$$x(t) = Ce^{at} = |C|e^{\theta t}e^{(r+jw_0)t} = |C|e^{rt}e^{j(w_0t+\theta)}$$
$$x(t) = |C|e^{rt}\cos(w_0t + \theta) + j|C|e^{rt}\sin(w_0t + \theta)$$

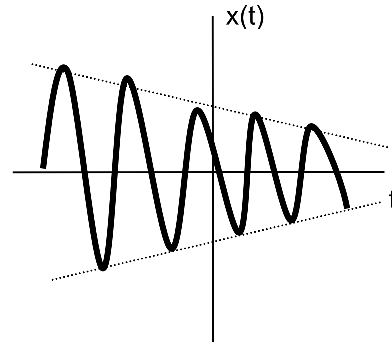
Below are the graphical representations of Real Part of x(t) (imaginary is ignored):



r=0 “*a is pure imaginary”*
$x(t)=|C|\cos(w_0t + \theta)$
Constant Sinusoidal Signal



r>0
$x(t)=|C|e^{rt}\cos(w_0t + \theta)$
Growing Sinusoidal Signal



r<0
$x(t)=|C|e^{rt}\cos(w_0t + \theta)$
Decaying Sinusoid Signal
damped sinusoid- RLC Circuit

- Example – Complex Exponential Signals
  Given the ability to generate complex exponential signal, $x(t)=Ce^{at}$, show how you can generated cos(wt) signal.
  Note: a and C are complex numbers and you may utilize Euler’s Identity.

  Solution:

❖ Discrete-Time Complex Exponential Sinusoidal Signals
A similar process that was applied to Continuous-Time may be used to explore Discrete-Time Complex Exponential Sinusoidal Signals. The general form of complex exponential signals in Discrete-Time is defined by:

$x[n]=Ca^n$ Where C and a are both complex numbers in the general case

$$a = |a|e^{j\phi_a} = a_{real} + ja_{imag}$$

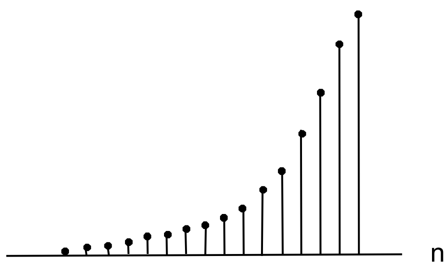
$$C = |C|e^{j\phi_c} = C_{real} + jC_{imag}$$

Note this equation is similar to Continuous-Time. It may be easier to see the similarity if you replace a by e^B which results in $x[n]=Ce^{Bn}$

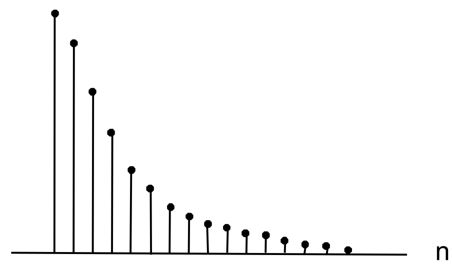
In the simplest case, when “a” and “C” are both real the complex exponential signal will become:

$x[n]=Ca^n$ *Real Exponential Signals*

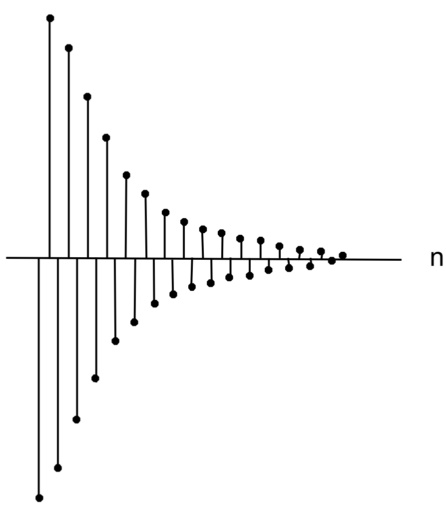
There are four types of Real Exponential Signals based on the values of “a” as shown below:



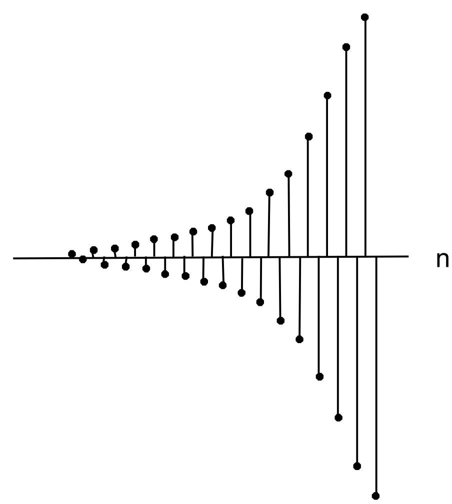
X[n] when a>1
signal grows exponentially



X[n] when 0<a<1
signal decays exponentially



X[n] when -1<a<0
signal decays exponentially, sign alternates



X[n] when a<-1
signal grows exponentially, sign alternates

Sinusoidal Signals are the next subclass that is deemed useful. This subclass is derived when |a|=1 and B is pure imaginary ($B=jw_o n$) as shown below:

$x[n] = Ce^{jw_0 n}$ *Sinusoidal Signals*

Using Euler's identity, we can rewrite the above equation in terms of complex exponentials by:

$$A\cos(w_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{jw_0 n} + \frac{A}{2} e^{-j\phi} e^{-jw_0 n}$$

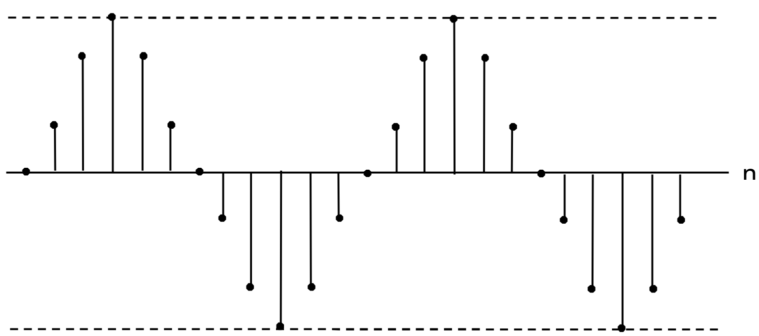
Now, we will is the time to explore general form of Complex Exponential Periodic Signals in Discrete-Time. First, let's rewrite the General Complex Exponential Signals in order to interpret them in-terms of real exponential and sinusoidal signals.

Apply the polar form of C and a to X[n] general form equation:

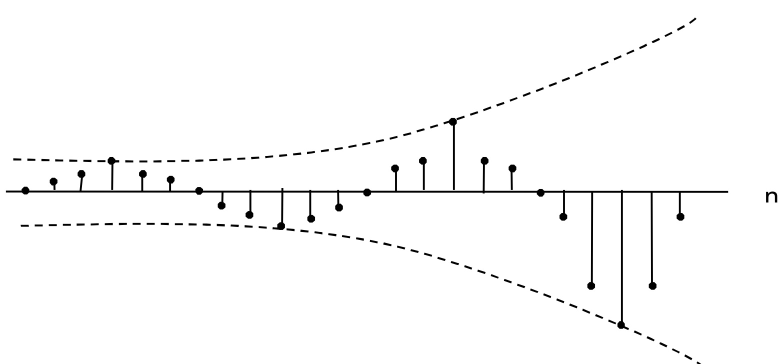
$$C = |C| e^{j\theta} \quad \& \quad a = |a| e^{jw_0}$$

$$x[n] = Ca^n = |C||a|^n \cos(w_0 n + \theta) + j|C||a|^n \sin(w_0 n + \theta)$$

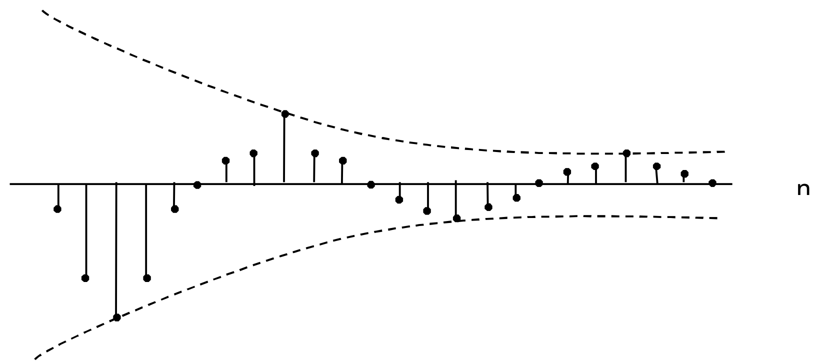
The above general form may be represented by one of the following three graphs based on the range of "a" values.



x(t) when |a| = 1 "Real & imaginary part of complex exponential are sinusoidal"



x(t) when |a| > 1 *"Sinusoidal multiplied by growing exponentials"*



x(t) when |a| < 1 *"Sinusoidal multiplied by decaying exponentials"*

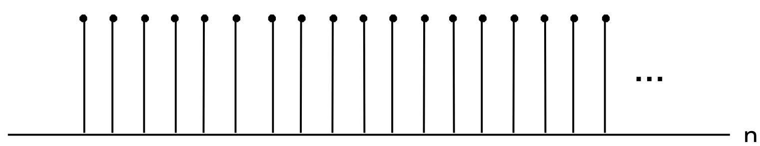
- Discrete-Time has similar periodic properties as Continuous-Time, but with same important distinctions. The exponential signal with frequency w_o is the same as any signal with frequencies ($w_o + 2k\pi$) where k is an integer:

  $e^{j(w_0+2\pi)n} = e^{jw_0 n}e^{j2\pi n} = e^{jw_0 n}$ Real part or Cos(w_0n) is typically plotted.

  Although any 2π period may be used as the period, commonly intervals include $0 \le w_0 \le 2\pi$ or $-\pi \le w_0 \le \pi$.

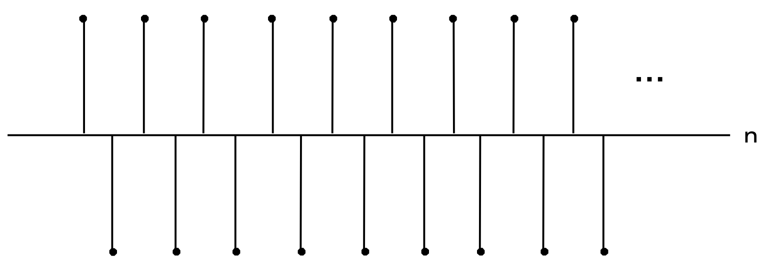
  Additionally consider the following:

  - w_o at $2n\pi$ and 0 which produce the lowest rate of oscillation:
    $e^{j2\pi} = e^{j0} = 1$



N =1 sample Period
$w_0 = 2\pi/N = 2\pi$
x[n]=cos(2πn)=cos(0)

  - w_o at $n\pi$ produces the highest number of oscillations $e^{jn\pi} = (-1)^n$



N =2 sample Period
$w_0 = 2\pi/N = \pi$
x[n]=cos(πn)

  - x[n]= $e^{jw_0 n}$ is periodic with period N only if w_oN is a multiple of 2π.
    It is important to note that unlike Continuous-Time, w_oN is not guaranteed to be a multiple of

2π. Therefore we need to ensure it is true when analyzing a signal.

Starting with the definition of periodicity. In order for x[n] to be periodic with period N, it must satisfy:

$e^{jw_0 n} = e^{jw_0(n+N)}$ For this equation to be true,

equation $e^{jw_0 N} = 1 = e^{j2\pi m}$ must hold true

Which means $w_0 N = 2\pi m$

therefore

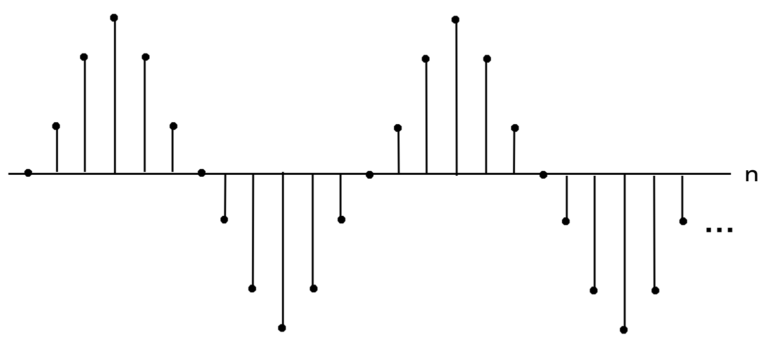
fundamental frequency. $w_0 = \dfrac{2\pi m}{N}$

fundamental period is $N = m(\dfrac{2\pi}{w_0})$

when integers m and N have no common factors

Finally, note that $\dfrac{w_0}{2\pi}$ must be a rational number for signal $e^{jw_0 n}$ to be periodic.

- Example – Fundamental Period and Frequency
  - Example – Find the fundamental period and frequency of the following Discrete-Time signals:



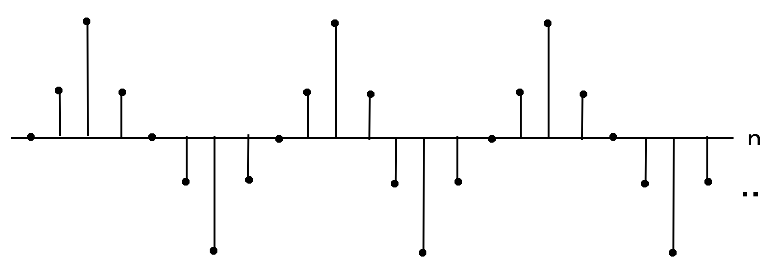
Solution:

N =12 Sample Period

$w_0 = 2\pi/N = \pi/6$

$x[n]=\cos(\pi n/6)$

  - Example – Find the fundamental period and frequency of the following Discrete-Time signals:



Solution:

N =8 Sample Period

$w_0 = 2\pi/N = \pi/4$

x[n]=cos(πn/4)

- Example – What is the fundamental frequency and period of the combined Discrete-Time signals represented by the following equation:
  $x[n] = e^{j(4\pi/8)n} + e^{j(2\pi/6)n}$

  Solution:

  First term has a fundamental period N_1=4
  Second term has a fundamental period N_2=6

  x[n] fundamental period, N, is the lowest common multiplier by N_1 & N_2
  N = 12 (evenly divisible to both terms fundamental period)

❖ Summary comparisons of Continuous-Time x(t)= $e^{jw_0 t}$ and Discrete-Time x[n]= $e^{jw_0 n}$

Continuous-Time x(t)= $e^{jw_0 t}$	**Discrete-Time x[n]=** $e^{jw_0 n}$
Each distinct value of w_0 results in unique signals	w_0 values separated by 2π are identical signals
Signal is periodic for all values of $w_0 = 2\pi/T$	Signal is periodic only if period $N = 2\pi m/ w_0$ is a postive integer
Fundamental frequency w_0	$w_0 = 2\pi m/N$ is fundamental frequency where *m is smallest value that results in integer N.*

➢ Example – Draw the following signals and find the period (if periodic) for the following signals:

a) $x[n] = \cos(2\pi n/12)$
b) $x(t) = \cos(2\pi t/12)$
c) $x[n] = \cos(8\pi n/27)$
d) $x(t) = \cos(8\pi t/27)$
e) $x[n] = \cos[n/6]$
f) $x(t) = \cos(t/6)$

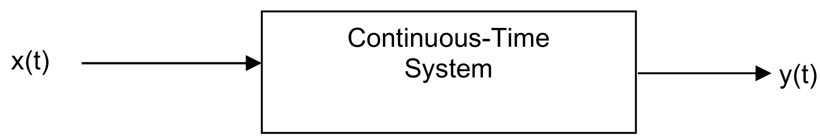
Solution
a) Periodic with period N=12
b) Periodic with period T=12

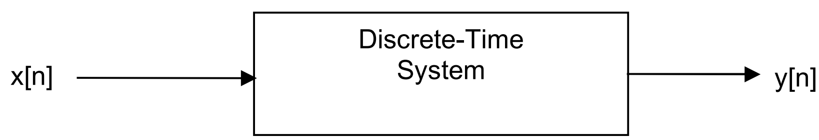
Remaining parts are to be completed by students.

1.7. Fundamental System Properties

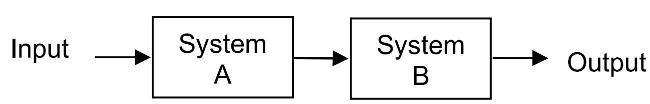
As was discussed earlier the concept of signals and systems applies to a broad range of fields beyond Electrical Engineering. The power of signals and systems comes from the fact that all problems to be solved or analyzed can be modeled as a system.

A system is a mathematical model of physical systems. It additionally describes the transformation from input to output signal. A system diagram shows system input and output. Below are examples of Discrete-Time and Continuous-Time system diagram:

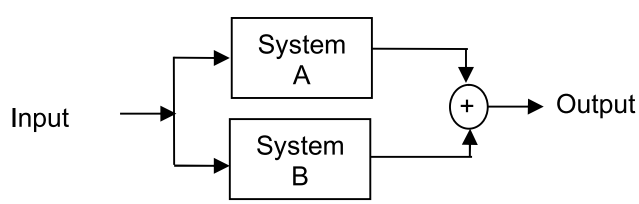




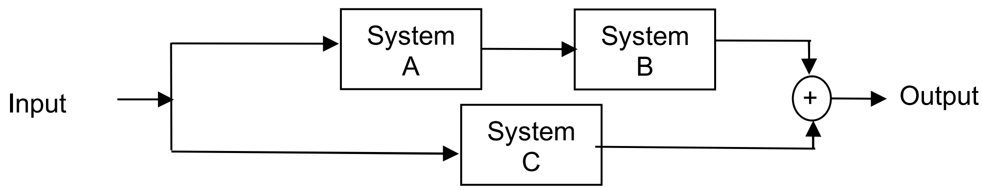
More complex systems can be designed by interconnecting simpler systems in order to perform more complex tasks. Systems are typically interconnected in one of the following four configurations:



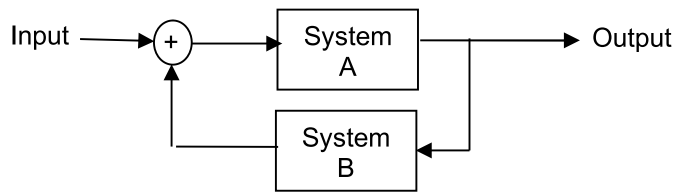
Cascade (Series) Interconnection



Parallel Interconnection



Series-parallel Interconnection



Feedback Interconnection

There are six fundamental system properties which are discussed here. These properties will play a crucial role in the development of additional tools used to analyze and design systems.

The six properties are Memory, Invariability, Causality, Stability, Time Invariance and Linearity. The rest of this section will explore each of these properties in more detail.

❖ Memory

A system with the ability to retain information is referred to as a system with memory. For example capacitors and inductors can be considered systems with memory. Resistor is considered memory-less because it does not have the capacity to retain information and only depends on the present input.

Mathematically, the distinction is made by the following definition of memory-less system and system with memory:

- Memory-Less System

  In a memory-less system, output only depends on present input, x(t). Below are a few examples of Memory-Less systems:

  (a) $y[n] = 20x[n] - x^4[n]$
  (b) $y(t) = Rx(t)$
  (c) The simplest memory-less system is the identity system where output is identical to input $y(t)=x(t)$ or $y[n]=x[n]$

- System with Memory

  In a system with memory, output depends on either past or future input. Below are examples of systems with memory:

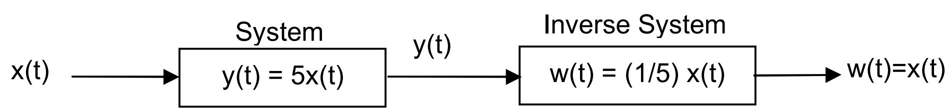
  (a) $y[n] = x[n+1]$
  (b) $y(t) = x(t) + 5x(t-3)$
  (c) $y[n] = \sum_{k=-\infty}^{n} x[k]$
  (d) $y(t) = \frac{1}{k}\int_{-\infty}^{t} x(\tau)d\tau$

  The above equation represents a capacitor if y(t)=v(t), x(t)=i(t) and k=C or the *equation represents an inductor if y(t)=i(t), x(t)=v(t) and k=L.*

- Invertibility

  A system is invertible if the system generates unique output in response to a unique input. In this type of system, output may be used to uniquely identify input, which is the benefit of Invertibility.

  In an invertible system, an inverse system exists such that when cascaded with the original system, the output of the combined system, w[n] is the same as the input x[n]



  For example y[n]=3 is not invertible since there is no way to identify a unique input. On the other hand y(t) = 4x(t) is invertible since input is uniquely related to the output.

  The system described by $y(t)=x^2(t)$ is not invertible since the sign of input is lost. Therefore input cannot be uniquely identified based on output.

  The concept of Invertibility is key to any system that takes an input which needs to be recovered at some point in the future. For example cell phones and computers.

- Causality

  A system is causal if the output depends only on present and past input. Causal systems do not anticipate input which is the reason it is also called a non- anticipative system. On the flip side, a non-causal system depends on future input.

  - Example - Is the system described by y[n]=x[n]+x[n+3] Causal?
    Solution:

    The system is non-causal since it depends on x[n+3] which is a future input.

  - Example - Is the system described by y[n]= x[-n] Causal?
    Solution:

    Before answering try n=-2 → note that y[-2]=x[2] is Non-causal

  - Example - Is the system described by y(t)=x(t)sin(t+2) Causal?
    Solution:

    Yes, since it only depends on current x(t). Note that sin(t+2) is just a factor and has nothing to do with the input.

  The concept of causality is a valuable concept in speech and image processing as well as geographical/meteorological signals.

- Stability

  A stable system is one in which small inputs lead to responses that do not diverge. An example of a stable physical system is a ball sitting at the bottom of an inverted cone. On the other hand, if the ball is balanced on top of a cone, it is unstable since the slightest force will dislodge the ball from the top. In general, stability of physical systems results from the presence of mechanisms which dissipate energy.

  A system is said to be stable if the system output is bounded in response to all bounded input. In other words, if the input is finite ($|x(t)|<\infty$) then the output will also be finite.

- Example - Is the system described by $y(t)=tx(t)$ stable?
  Solution:
  The system is unstable since for bounded $x(t) \rightarrow y(t)$ is unbounded as t approaches infinity.

- Example - Is the system described by $y(t)= e^{x(t)}$ stable?
  Solution:
  The system is stable since $y(t)$ is bounded as long as $x(t)$ is bounded.

- Time Invariance
  A system is time invariant if the system characteristic does not change over time. For example a typical RLC circuit is time invariant since its behavior does not change from one minute to the next.

  The formal definition is that a system is time invariant if an input shift-in-time results in an identical time-shift in the output.

  - Example - Is the system described by $y(t)=5\cos[x(t)]$ time invariant?
    Solution:
    First apply the input $x_1(t)$ to the system $\rightarrow$
    $y_1(t)=5\cos[x_1(t)]$ and shift by $t_0 \rightarrow y_1(t-t_0)=5\cos[x_1(t-t_0)]$
    Second apply the time shifted input $x_2(t) = x_1(t-t_0)$ to the system $\rightarrow$
    $y_2(t)=5\cos[x_1(t-t_0)]$

    . Since $y_1(t- t_0) = y_2(t)$ then the system is time invariant.

  - Example: Is the system described by $y[n]=2nx[n]$ time invariant?
    Solution:
    Using the formal approach…

    First apply the input $x_1[n]$ to the system $\rightarrow$
    $y_1[n]=2nx[n]$ and shift by $n_0 \rightarrow y_1[n-n_0]=2(n-n_0)x_1[n-n_0]$
    Second apply the time shifted input $x_2[n] = x_1[n-n_0]$ to the system $\rightarrow$
    $y_2(t)=2n\, x_1[n-n_0]$

    . Since $y_1(t- t_0) \neq y_2(t)$ then the system is not a time invariant system.

    Alternative approach - Sometimes it is easier to find one input that violates the time invariant rule such as is done below:

    1) set $x_1[n]=\delta[n] \rightarrow y_1[n]=n\delta[n]=0$ always (Hint: $\delta[n]=1$ only if $n=0$)
    2) set $x_2[n]= \delta[n-1] \rightarrow y_2[n]=n\delta[n-1]= \delta[n-1]$

    Since $y_2[n]$ from shifted input is not equal to shift output $y_1[n-1]$ then this is not a time invariant system.

  - Example: Is the system described by $y(t)=x(t + 2)$ time invariant?
    Solution:

❖ Linearity

A linear system exhibits the superposition property which defines linearity. Superposition has the following two characteristics:

1) Additive Characteristic → response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$
1) Homogeneity Characteristic → response to $ax_1(t)$ is $ay_1(t)$ *where a is any complex constant*

Both of the characteristics can be combined into a single statement which may be stated for CT and DT as shown below:

Continuous-Time (CT): $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
Discrete-Time (DT): $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$
Note: a and b can be complex numbers.

The Superposition property statements can be generalized to:

Continuous-Time (CT): $$x(t) = \sum_k a_k x_k(t) \rightarrow y(t) = \sum_k a_k y_k(t)$$

Discrete-Time (DT): $$x[n] = \sum_k a_k x_k[n] \rightarrow y[n] = \sum_k a_k y_k[n]$$

➢ Example - Is the system described by $y(t)=tx(t)$ linear?

Note
if $y_3(t) = ay_1(t) + by_2(t)$
Then “a linear system”
Else “Not a linear system”

Solution
Apply $x_1(t) \rightarrow y_1(t)=tx_1(t)$
Apply $x_2(t) \rightarrow y_2(t)=tx_2(t)$

$x_3(t)= ax_1(t) + bx_2(t) \rightarrow y_3(t)=t\{ax_1(t) + bx_2(t)\}$

Since $y_3(t) = ay_1(t) + by_2(t)$, This is a linear system.

➢ Example - Define the properties of a system where x(t) is the input and y(t) is the output as defined below:
a) $y(t) = 2t + 1$
b) $y(t) = 2x(t) +1$
c) $y(t) = t^2 + x(t+1)$
d) $y(t) = (t-1)x(t)$

Solution
Student Exercise.

- Example - Define the properties of a system where x[n] is the input and y[n] is the output as defined below:
  - a) y[n] = x[n]
  - b) y[n]=x[n+1]
  - c) y[n] = x[n]x[n-2]
  - d) $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$

  Solution

  Student Exercise.

- Example - Define the properties of a system where x[n] is the input and y[n] is the output as defined below:

  $y[n] = \cos[w_0 n + \theta]\ x[-n + 3]\{u[-5 + n] – u[n - 9]\}$

  Solution

  Student Exercise.

1.8. Statistical Properties of Noise

Discussion regarding signal and system is not complete without an overview of noise in the system. As the name implies, noise is an unplanned, unwanted and in many cases an unknown quantity in the system. Noise may have been introduced to the system from other devices, natural phenomenon or the system itself. The goal of a designer is to be able to separate the signal which carries the information even in a noisy environment.

In cases where noise is understood and has a well defined characteristic, then it may be filtered out however when the noise is distributed over a range and has a random nature then a broad solution is needed based on the profile of the noise. The most common approach to understanding and treatment this noise is through probabilities and statistics.

In this section we will introduce one of the most common noise classes which is white noise. White noise is a random signal which is distributed with a uniform, Gaussian or other probability distributions. Before discussing an example of white noise distribution, let's start with the definition of probability distribution.

In statistics, a probability distribution is defined as:

- The probability of occurrence of any value of an unidentified random variable for Discreet-Time.
- The probability of occurrence of any value falling within a particular interval for Continuous-time.

The most common type of white noise is the uniformly distributed noise; random signal with a uniformly distributed value between two frequencies.

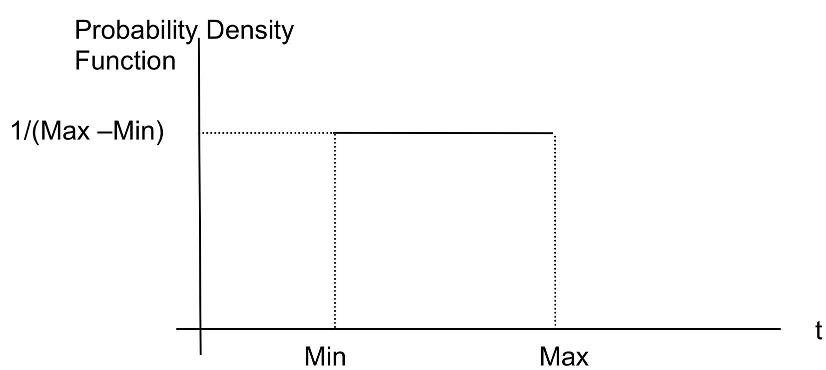
- Discrete Uniform Distribution

  Discrete Uniform Distribution is a discrete probability distribution that allows all values of a finite set equal probability of occurring. Let's say the set of possibilities are $A \in \{A_1, A_2, \ldots A_n)$ and they are distribution uniformally. At any given point, each possibility (A_i) has probability (1/n) to be present.

  In MATLAB such a discrete uniformally distributed white noise may be created by the use of a random number generator that is limited to the list of possibilities.

- Continuous Uniform Distribution

  Continuous Uniform Distribution is a probability distribution where each value in a given range has equal probability of occurring. Below is the probability function:



$$p(t) = \begin{cases} \dfrac{1}{Max - Min} & when \quad Min \le t \le Max \\ 0 & when \quad t < Min \quad or \quad t > Max \end{cases}$$

The uniformally distributed white noise plays an important role since it allows the testing of design to verify that the signal can be recovered when noise is present at a given range of frequencies. The random nature forces designers to find solutions other than filters which are static in their basic implementation.

1.9. Chapter Summary

This section is a summary of key concepts from this chapter.

- Power and Energy of signal x(t)
  - Continuous-Time Signal x(t) where |x(t)| is the amplitude of Complex value x(t)
    - Total energy over the time interval $t_1 \le t \le t_2$ → $E = \int_{t_1}^{t_2} |x(t)|^2 \; dt$
    - Average power over the time interval $t_1 \le t \le t_2$ → $P = \dfrac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \; dt$
    - Energy and Average Power in interval $-\infty \le t \le \infty$

      $E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$ and $P_\infty = \lim_{T \to \infty} \dfrac{1}{2T} \int_{-T}^{T} |x(t)|^2 \; dt$
  - Discrete-Time Signal x[n] where |x[n]| is the amplitude of Complex value x[n]
    - The total energy over the time interval $n_1 \le n \le n_2$ → $E = \sum_{n=n_1}^{n_2} |x[n]|^2$
    - The average power over the time interval $n_1 \le n \le n_2$ → $P = \dfrac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$
    - Energy and Average Power in interval $-\infty \le n \le \infty$

      $E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$ and $P_\infty = \lim_{T \to \infty} \dfrac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

- Independent Variable Transformation
  - Time Shift
    - x(t) →x(t – t_0)
    - x[n] →x[n – n_0]
  - Time Reversal
    - x(t) →x(-t)
    - x[n] →x[-n]
  - Time Scaling
    - x(t) →x(at)
    - x[n] →x[an]
  - General Form
    - x(t) →x(at - b)
    - x[n] →x[an - b]

- Symmetry
  - Pure Odd Signals
    - x(t) = -x(-t)
    - x[n] = -x[-n]
  - Pure Even Signals
    - x(t) = x(-t)

- x[n] = x[-n

- Non-pure Signal
  - Even part {x(t)} = ½{x(t) + x(-t)}   or   {x[n]} = ½{x[n] + x[-n]}
  - Odd part of {x(t)} = ½{x(t) - x(-t)}   or   {x[n]} = ½{x[n] - x[-n]}

- Complex Exponential and Sinusoidal Signals
  - Continuous-Time

$x(t) = Ce^{at}$ Where C and a are both complex numbers

$$a = |a| e^{j\phi_a} = a_{real} + ja_{imag}$$

$$C = |C| e^{j\phi_c} = C_{real} + jC_{imag}$$

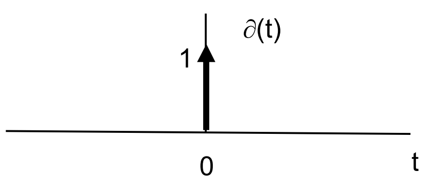
  - Discrete-time

$x[n]=Ca^n$ Where C and a are both complex numbers in general case

$$a = |a| e^{j\phi_a} = a_{real} + ja_{imag}$$

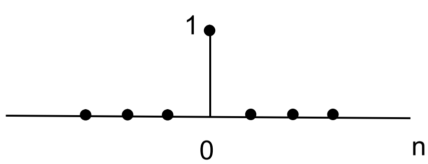
$$C = |C| e^{j\phi_c} = C_{real} + jC_{imag}$$

- Unit Impulse Function
  - Unit Impulse, ∂(t)



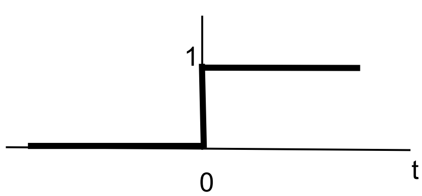
$$\partial(t) \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

  - Unit Impulse, ∂[n]



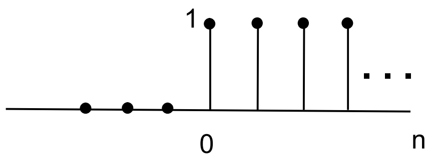
$$\delta\,[n] \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

- Unit Step Function
  - Unit Step, u(t)



$$u[t] \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

- Unit Step, u[n]



$$u[n]\begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

- System Properties
  - Memory-less System
    In a memory-less system, output only depends on present input.
  - Invertibility
    A system is invertible if the system generates unique output in response to a unique input.
  - Causality
    A system is causal if the output depends only on present and past input.
  - Stability
    A Stable system is one in which small inputs lead to responses that do not diverge.
  - Time Invariance
    A system is time invariant if the system characteristic does not change over time.
  - Linearity
    A linear system exhibits the superposition property which defines linearity.

1.10. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 1.

- Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 2

- Stremler, F. Introduction to Communication Systems (1990) Addison-Wesley Publishing Company
  Chapt 2

- Introduction to Statistics

1.11. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 2. Linear Time-Invariant (LTI) Systems

Key Concepts and Overview

- Linear Time Invariant (LTI) System Overview
- Convolution Sum in Discrete-Time LTI Systems
- Convolution Integral in Continuous-Time LTI Systems:
- LTI Systems Properties
- Differential/Difference Equations
- Additional Resources

2.1. Linear Time Invariant (LTI) System Overview

Fundamentals of signals and systems were introduced in previous chapter. In this chapter, the focus is on defining Linear Time Invariant (LTI) systems which serve as the start point for modeling systems. LTI systems are also the foundation for the topics covered in the next chapters. Linear Time Invariant systems as the name implies have two defining properties:

- Linearity
  In a linear system superposition holds true which means responses to individual input can be summed up to calculate the total system response.

- Time-Invariance
  Time-Invariance provides for shifting input in time and expecting output to shift in time without changing its characteristic.

The following sections explore the concepts of system impulse response, convolution sum/integration, and difference/differential equations for both the Discrete-Time and Continuous-Time Linear Time Invariant (LTI) systems.

2.2. Convolution Sum in Discrete-Time LTI Systems

Any signal may be described using impulse functions as discussed in the previous chapter. In general any signal can be written as:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

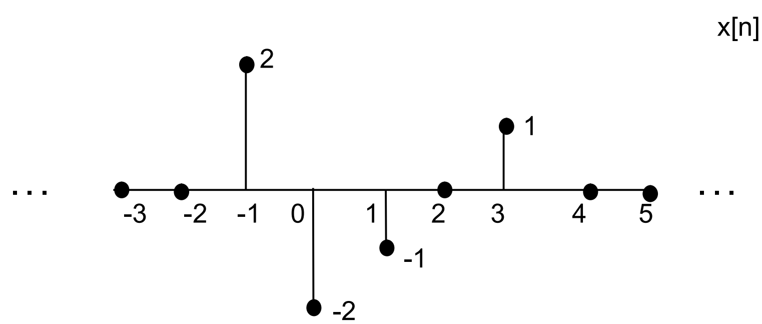
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This equation is built based on the fact that impulse functions can be used to isolate (or sample) a single value of the signal out of the total signal. This feature is referred to as the sifting property of the Unit Impulse function. Again, the Unit Impulse function δ[n-k] is non-zero only when k=n, so it sifts through the signal for value of x[k]. Also, step function u[n] is related to impulse function as it is described by the following equation:

$$u[n] = \sum_{k=0}^{k=\infty} \delta[n-k]$$

- ❖ Examples – Use of Impulse Function to Describe a Discrete-Time Signal

  - ➢ Example – Use the sifting property of impulse response functions to mathematically describe the function described graphically by the following:



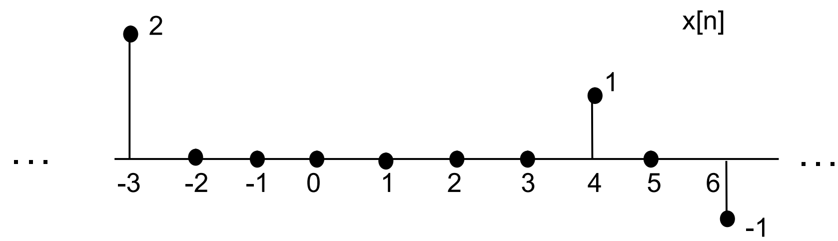
Solution:

$$x[n] = 2\delta[n+1] - 2\delta[n] - \delta[n-1] + \delta[n-3]$$

  - ➢ Example – Plot the function described by the following equation:

$$x[n] = 2\delta[n+3] + \sum_{k=2}^{3} (-1)^k \delta[n-2k]$$

Solution:



- Unit Impulse Response

  LTI systems may be characterized by their Unit Impulse response or simply impulse response, h[n]. Impulse response is the superposition of responses of the LTI system to impulse function from negative to positive infinity. The following diagram shows the relationship among the impulse response, input and output:

$$x[n]=\sum_{k=-\infty}^{k=\infty}\delta[n-k] \longrightarrow \boxed{\begin{array}{c}\textbf{h[n]}\\ \text{Impulse Response}\end{array}} \longrightarrow \text{y[n]=h[n]}$$

  Another way of stating the same concept is that the response of a linear system to x[n] is the superposition of the scaled system responses to x[n]. This can be represented by:

  h[n-k]=aδ[n-k] when n=k

  The next section leverages this finding to calculate output based on input and impulse response.

- The Convolution Sum

  The convolution sum is a method of finding output, y[n], of a system in response to an input signal x[n] that is not the impulse function, δ[n]. This is done with the use of the convolution sum and the impulse response function, h[n]. Conceptually, each non-zero value of input should be multiplied by each non-zero value of h[n] and summed to calculate the value of the output. This is implemented by the following equation which is referred to as the convolution sum:

$$y[n]=\sum_{k=-\infty}^{k=\infty}x[k]h[n-k]$$ "The Convolution Sum and corresponding notation"

$$y[n]=x[n]*h[n]$$

  Another useful tool in understanding and applying the convolution sum is the graphical approach. The following steps can be used to graphically calculate the output using the convolution sum:

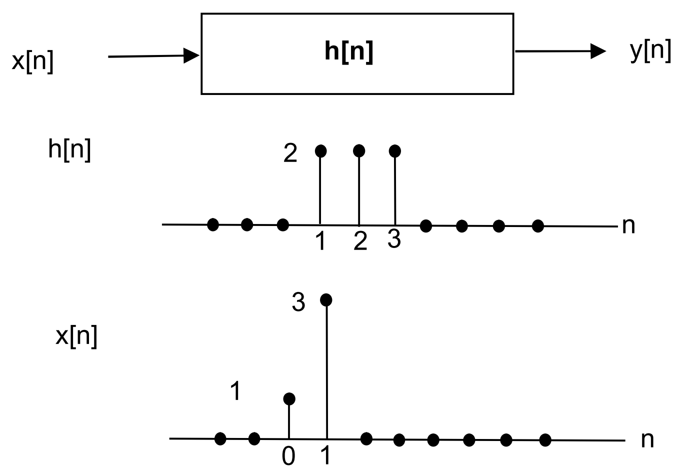
  (1) Draw the signal h[n-k]
  * Reflect h[k] about the k=0 axis which results in h[-k]
  * Shift h[n-k] from -∞ to +∞ by changing n from -∞ to +∞.
  Identify intervals where h[n-k] & x[k] hold their function and neither is zero.

  (2) Calculate the product of x[n] and h[n-k] from -∞ to +∞
  Note: There may be many intervals that the products are zero.

  (3) Sum the product of h[n-k] and x(k)

❖ Examples – The Convolution Sum

➢ Example – For the following Linear Time Invariant (LTI) system, use the convolution sum equation and graphical method to find y[n]:

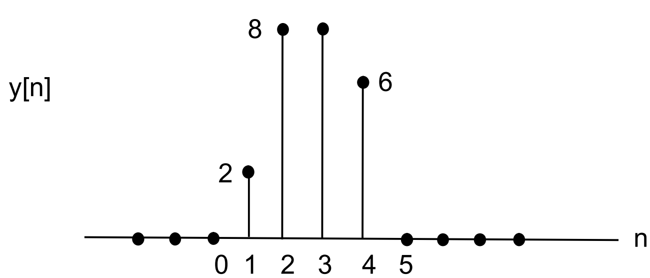


Solution:

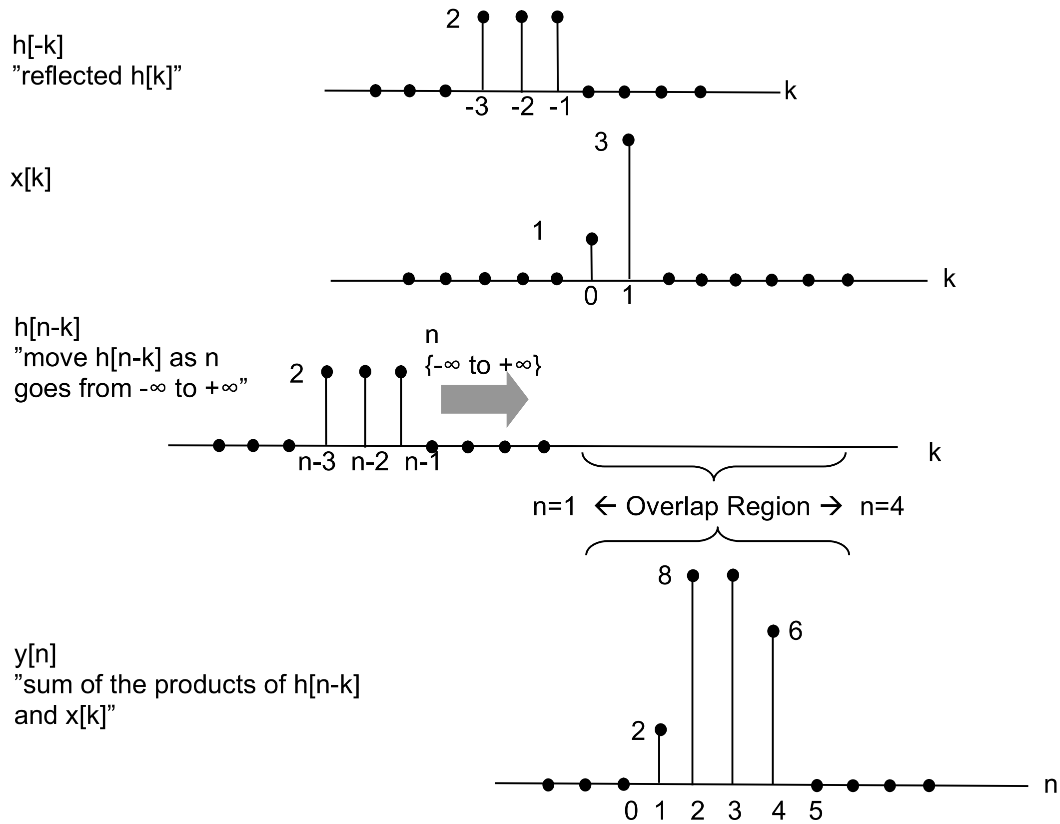
Using the equation to calculate y[n]

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$$

Note: only at k=0 and 1, input is non-zero therefore y[n] is:
y[n]=x[0]h[n] + x[1]h[n-1] = h[n] + 3h[n-1]
Furthermore h is only non-zero at n=1,2 and 3 therefore:



Using Graphical application of the Convolution Sum to calculate y[n]

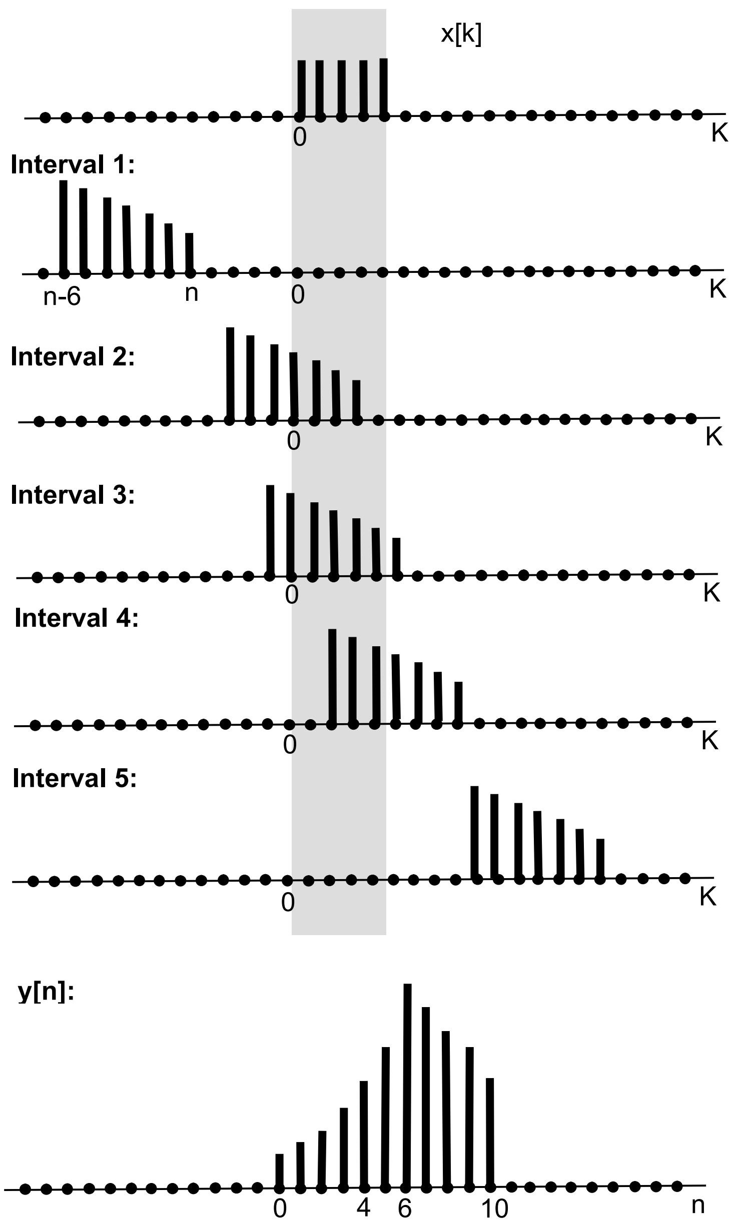


- Example - Find the value of y[n] by applying the convolution sum graphically to the following LTI system:

x[n] = $\begin{matrix} 1 & 0 \le n \le 4 \\ 0 & otherwise \end{matrix}$ & h[n]= $\begin{matrix} a^n & 0 \le n \le 6 \\ 0 & otherwise \end{matrix}$

Solution:

Apply $y[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$ graphically



h[n-k] for n<0 → no overlap

$$y[n] = \sum_{k=-\infty}^{0} x[k]h[n-k] = 0$$

h[n-k] for $0 \le n \le 4$ → partial right Overlap

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] \quad 0 \le k \le 4$$

h[n-k] for $4 < n \le 6$ → full Overlap

$$y[n] = \sum_{k=0}^{4} x[k]h[n-k] \quad 0 \le k \le 4$$

h[n-k] for $6 < n \le 10$ → partial left Overlap

$$y[n] = \sum_{k=(n-6)}^{4} x[k]h[n-k] \quad (n-6) \le k \le 4$$

h[n-k] for $n > 10$ → no Overlap

$$y[n] = 0$$

Y[n]

= 0 for n < 0

$= \sum_{k=0}^{n} a^{n-k} = \dfrac{1-a^{n+1}}{1-a}$ $for\ 0 \le n \le 4$

$= \sum_{k=4}^{6} a^{n-k} = \dfrac{a^{n-4}-a^{n+1}}{1-a}$ $for\ 4 < n \le 6$

$= \sum_{k=n-6}^{4} a^{n-k} = \dfrac{a^{n-4}-a^{7}}{1-a}$ $for\ 6 < n \le 10$

= 0 for n > 10

❖ Example – Find the value of y[n] by applying the convolution sum to the following LTI system and input:

x[n] = $e^{-3n}\{u[n-5]+u[n-2]-2u[n-6]\}$ → h[n] = $3^n\,\delta[n]+\delta[-n+2]$ → y[n]

<u>Solution:</u>

Student Exercise

❖ Example – Find and plot y[n] for an LTI system where:

Input, x[n] = 20 sin(0.005πn){u[n-3]-u[n-9]}

Impulse response, h[n]= $10e^{-n}\{\partial[n+1] + 3\partial[n+10]\}$

<u>Solution:</u>

Student Exercise

2.3. Sidebar Notes (Useful Relationships)

Geometric Series are useful in simplifying the results of the Convolution Sum:

1) Finite Geometric Series:

$$\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r}$$

2) Infinite Geometric Series (n=∞) when |r|<1:

$$\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$$

3) Infinite Geometric Series (n=∞) when |r|<1 and m=0:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

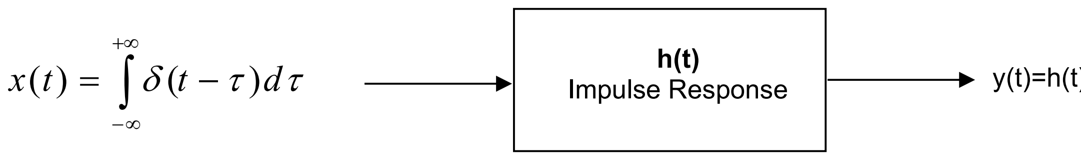
The Geometric Series may be used to derive more complex series such as:

$$\sum_{k=0}^{\infty} \frac{\sin(kx)}{r^k} = \frac{r\sin(x)}{1-r^2-2r\cos(x)}$$

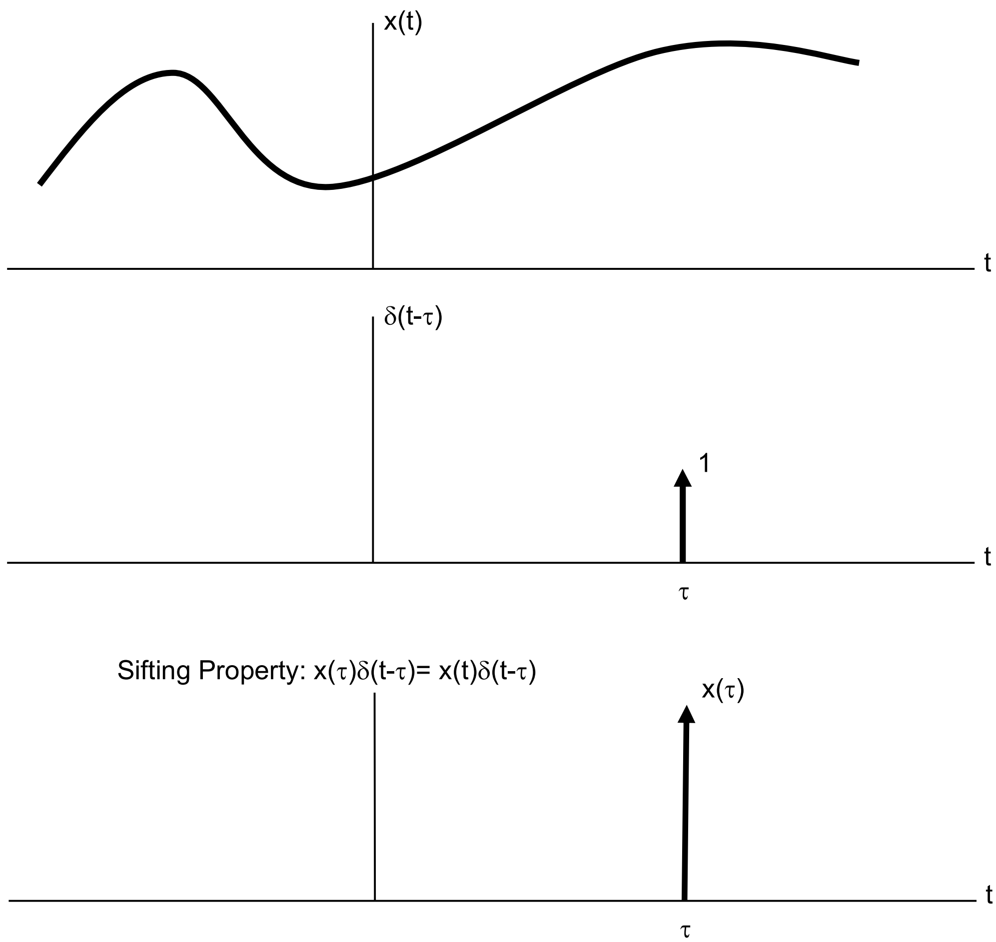
2.4. Convolution Integral in Continuous-Time LTI Systems

The convolution integral approach to calculating output in Continuous-Time LTI system is similar to the one introduced in the previous section for the Discrete-Time. The main difference is that the interval between two adjacent points on the signal is infinitely small ($\Delta t \rightarrow 0$).

LTI systems may be characterized by their Unit Impulse response or simply impulse response, h(t). Impulse response is the superposition of responses of the LTI system to impulse function from negative infinity to positive infinity. The following diagram shows the relationship amongst impulse response, input and output:

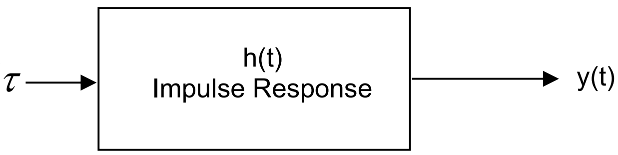


Similar to the Discrete-Time, Unit Impulse function's sifting property "$x(\tau)\delta(t-\tau)= x(t)\delta(t-\tau)$" can be used to represent a single value of the signal (sample) as shown below:



Extending the sifting property of impulse function, all of input signal, x(t), to the LTI system is represented by:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$



The Continuous-Time convolution integral is a method of finding system output, y(t), that is generated in response to an input signal x(t), using the impulse response function, h(t). It is important to note that x(t) is not necessarily just an impulse function, $\delta(t)$.

Conceptually, each non-zero value of input should be multiplied by each non-zero value of h[n] and summed to calculate the value of output. This is implemented by the following equation which is called the convolution integral:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

"The Convolution Integral and corresponding notation"

$$y(t) = x(t) * h(t)$$

Another useful tool for understanding and applying the convolution integral is the graphical approach. The following steps can be used to graphically calculate the output using the convolution integral:

(1) Draw the impulse response transformation, $h(t-\tau)$
   * Reflect $h(\tau)$about the k=0 axis which results in $h(-\tau)$
   * Shift $h(t-\tau)$from $-\infty$ to $+\infty$ by changing n from $-\infty$ to $+\infty$.
   Identify intervals where $h(t-\tau)$& $x(\tau)$ *hold their function and neither is zero.*

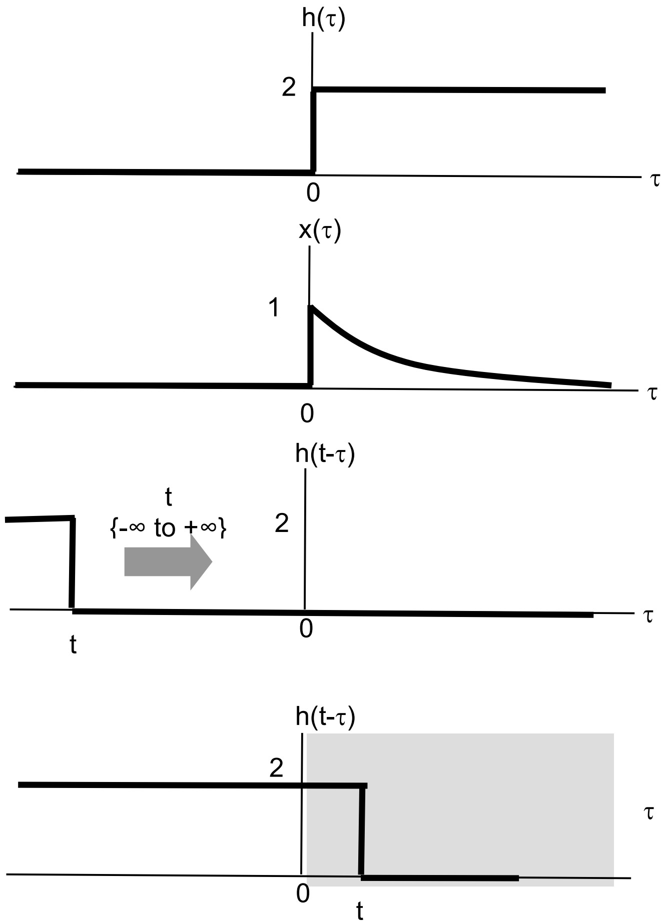
(2) Calculate the product of $x(\tau)$ and $h(t-\tau)$ from $-\infty$ to $+\infty$
   Note: There may be many intervals that the products are zero.

(3) Sum the product of $h(t-\tau)$ and $x(\tau)$

❖ Example – The Convolution Integral

➢ Example - Find y(t) for the Linear Time Invariant (LTI) system described by "h(t) = 2u(t)" when input is "x(t) = e^{-3t} u(t)".

Solution:



Interval 1: For t<0 → y(t)=0

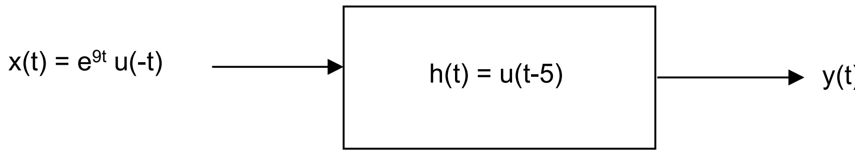
Interval 2: For t≥0

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

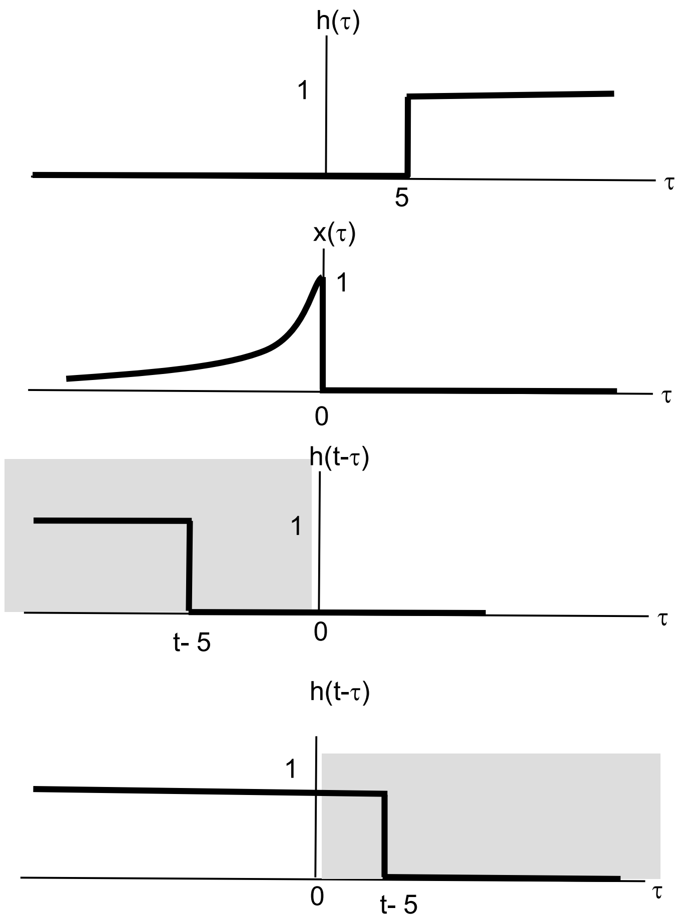
$$y(t) = \int_0^t 2e^{-3\tau}d\tau = -\frac{2}{3}(e^{-3t} - e^{-0})$$

$$y(t) = -\frac{2}{3}(e^{-3t} - 1)$$

- Example - Find y(t) for the Linear Time Invariant (LTI) system described by:



Solution



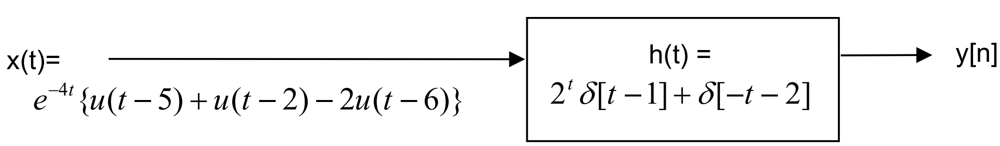
Interval 1: For (t- 5)≤0 → t≤5

$$y(t) = \int_{-\infty}^{t-5} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t-5} e^{9\tau}d\tau = \frac{1}{9}e^{9(t-5)}$$

Interval 2: For t -5 >0 → t>5

$$y(t) = \int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{0} e^{9\tau}d\tau = \frac{1}{9}$$

- Student Exercise - Find the value of y(t) by applying Convolution Sum to the following LTI system and input:



2.5. Linear Time-Invariant (LTI) Systems Properties

This section re-examines the system properties which were initially introduced in the previous section. In this section, the focus is on Linear Time-Invariant Systems and the convolution operations. The remainder of this section characterizes LTI systems in-terms of system's Impulse response, h(t):

- The Commutative Property
  The Commutative Property asserts that the order of convolution does not impact the results. The commutative property is represented by the following equations:

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

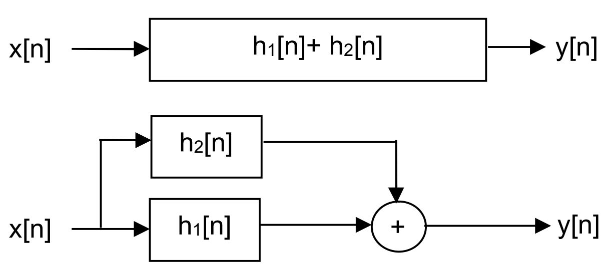
$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} h[n]x[n-k]$$

  The Commutative property may be proved by replacing "τ" with "t-p" in Continuous-Time or replacing "k" with "n-r" in discrete-time.

- The Distributive Property
  The convolution may be distributed over addition without affecting the results. The Distributive Property can be defined by:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

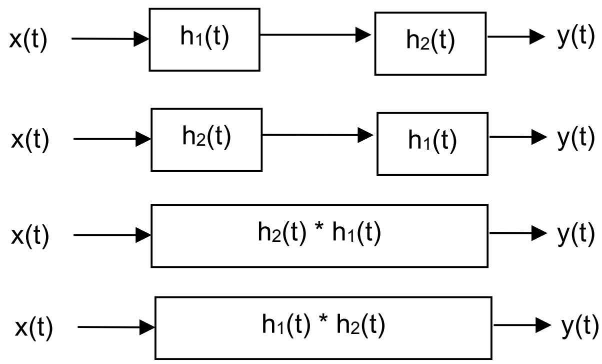
  The Commutative property may be proved by writing out the convolution sum/integral and factoring out the terms. The implication of the Commutative property is that both of the following configurations are equivalent:



- The Associative Property
  The Associative property states that the order of convolution does not affect the result. Below is the restatement of this property in the equation form:
  .

$$x(t) * [h_1(t) * h_2(t)] = x(t) * h_1(t) * h_2(t)$$
$$x[n] * (h_1[n] * h_2[n]) = x[n] * h_1[n] * h_2[n]$$

  The Associative property may be proved by writing out the sum/integral and factoring out the terms. The implication of the Associative property is that all of the following configurations are equivalent:



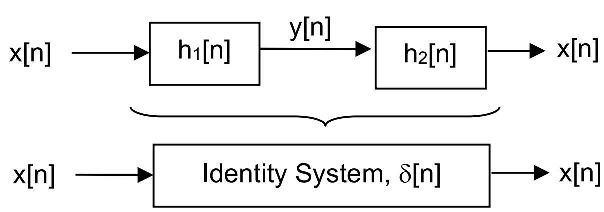
Of course, the same is true for Discrete-Time LTI system.

❖ LTI System with and without memory
Memory-less LTI system's output only depends on the value of present input. Therefore, memory-less system impulse response, h(t) or h[n], satisfies the following relationships:

$h[n]=0$ for $n\neq 0 \rightarrow h[n]=K\delta[n] \rightarrow y[n]=Kx[n]$ "Discrete-Time"
$h(t)=0$ for $t\neq 0 \rightarrow h(t)=K\delta(t) \rightarrow y(t)=Kx(t)$ "Continuous-Time"

All other systems are LTI systems with memory, by definition.

❖ Invertibility of LTI Systems
A system is invertible only if an inverse system and impulse response exist that when connected in series to the original system, produces an output equal to the input of the first system.



In order for Invertibility to be true, $h_1(t) * h_2(t)$ must be an identity function, $\delta(t)$ where $h_1(t)$ is the original system impulse response and $h_2(t)$ is the inverse system impulse response. The following equation restates the definition of an invertible system:

$h_1(t) * h_2(t) = \delta(t)$
$h_1[n] * h_2[n] = \delta[n]$

➢ Example -- A system's input and output are related by the following equation:

$y(t)=5x(t-4)$

Is this system invertible?

Solution:
We need the system impulse response in order to answer the question of invertibility. By inspecting the given input/output relationship, it is clear that the output is equal to the input shifted and scaled. Therefore:

$h_1(t) = (1/5)\,\delta(t - 4)$ "System Impulse Response"

System is invertible only If an inverse impulse response, $h_2(t)$, can be found such that

$h_1(t)*h_2(t) = \delta(t)$.

It can be seen that $h_1(t) = 5\,\delta(t + 4)$ would satisfy the invertibility requirement:

$h_1(t)*h_2(t)= (1/5)\,\delta(t-4)*(5)\delta(t+4)= \delta(t)$

- Student Exercise – Show that LTI system with impulse response $h_1[n]=u[-n+2]$ is invertible by calculating the inverse impulse response and testing it.
  Hint: $u[n]*(\delta[n] - \delta[n-1]) = \delta[n]$

- Causality of LTI Systems
  In a causal LTI system, the system's present output only depends on the present and past input. In other words, the system is non-causal if the system output depends on the future input.

  For a causal Discrete-Time LTI system, y[n] does not depend on x[k] where k > n. This requirement translates to the rule that in a causal system, h[n]=0 for n<0.

  Examine the Convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$, to relate the two requirements.

  Similar requirements for causality exists in Continuous-Time which states that h(t)=0 for t<0 in causal systems.

  In summary, a system is causal if its impulse response "h(t) or h{n]" is zero for "t<0 or n<0".

- Stability for LTI Systems
  A system is stable if for every bounded input, the output is also bounded. Below is a restatement of stability definition in-term of impulse response:

  - Discrete-Time
    Input is bounded → |x[n]| < A for all n where A is a finite value

    $|y[n]| = |\sum_{k=-\infty}^{+\infty} h[k]x[n-k]| \quad \Rightarrow \quad |y[n]| \le |A \sum_{k=-\infty}^{+\infty} |h[k]|$ for all n

    The above equation is true if:

    $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$ then the system is stable

  - Continuous-Time
    Using a similar process to Discrete-Time, we can conclude that if:

    $\int_{-\infty}^{+\infty} |h[\tau]|\, d\tau < \infty$ then the system is stable

Although impulse response is used commonly to characterize systems, LTI systems may also be characterized by the unit step response, s[n] or s(t). The unit step response, s[n] or s(t), is the output of the system when input is u[n] or u(t).

x[n] =u[n] → Unit Response, s[n] → y[n] =s[n]

x(t) =u(t) → Unit Response, s(t) → y(t)=s[(t)

Leveraging the relationship between impulse and unit function, we can derive the following relationships between impulse and unit response of a LTI system:

s[n] = u[n]*h[n] = $\sum_{k=-\infty}^{n} h[k]$ and h[n] = s[n] –s[n-1] "Discrete-Time"

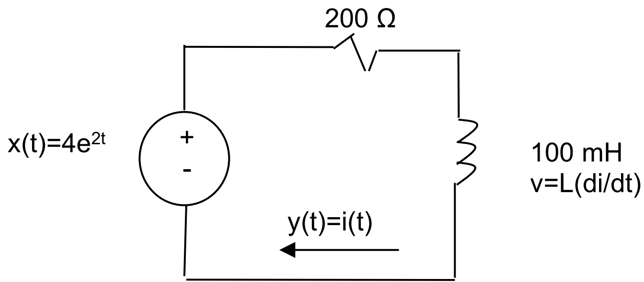
s(t) = u(t)*h(t) = $\int_{-\infty}^{t} [\tau] d\tau$ and h(t) = s'(t) = ds(t)/dt "Continuous-Time"

2.6. Differential/Difference Equations

Differential/Difference equations can be used to describe important subclasses of LTI systems. The linear constant-coefficient differential equation is used for Continuous-Time and the linear constant coefficient difference equation is used for discrete-time. Although these subclasses are more limited in scope, they serve as tools in analysis of an important subclass of systems which include the RC and RL circuits. Additionally, they serve as the basis for study of broader types of signals and systems. This section introduces and applies difference and differential equations with linear constant-coefficient.

❖ Linear Constant-Coefficient Differential Equations

First, let's use an RC circuit to derive the linear constant-coefficient differential equation. In the following LTI system, voltage is the input, x(t), and current is the system output, y(t)::



.Using KVL → $200y(t)+0.1\dfrac{dy(t)}{dt}=x(t)$

One difference is that in this equation y(t) is not given in-terms of x(t). We must solve the differential equation to find the output. The solution will have constants which would require additional relationships between input/output such as initial condition in order to find the constants' value.

Solution to an ordinary linear differential equation may be written as:

$y(t) = y_p(t) + y_h(t)$ where

$y_p(t)$ or the particular solution will be the same form as the input in this case $y_p(t)=Ae^{2t}$

$y_h(t)$ or the homogenous solution will have the form $y_h(t)=Be^{st}$ with $x(t)=0$

Taking advantage of the superposition property of LTI systems, we can find each portion of the solution (output or response) and then sum them to find the total solution.

First, plug $y_p(t)=Ae^{2t}$ (same as input except for the coefficient) into the differential equation to find the particular solution:

$$200Ae^{2t}+0.1\frac{d\{Ae^{2t}\}}{dt}=4e^{2t}$$

$$200A+0.2A=4 \Rightarrow A=4/200.2 \Rightarrow y_p(t)=0.02e^{2t}$$

Second, plug $y_h(t)=Be^{st}$ (power or coefficient are unknowns) into the differential equation where $x(t)=0$ to find the homogenous solution:

$$200Be^{st}+0.1\frac{d\{Be^{st}\}}{dt}=0 \qquad "x(t)=0 \textit{ for } \text{hom}\textit{ogeneous differntial equation}"$$

$$(200+0.2s)Be^{st}=0 \Rightarrow (200+0.2s)=0 \Rightarrow s=-0.01 \Rightarrow y_h(t)=Be^{-0.01t}$$

Therefore the total solutions is : $y(t)= y_p(t)+ y_h(t)= 0.02e^{2t} + Be^{-0.01t}$

Now, we need to find the coefficient B by using the initial state {if not given, assume $y(t_0)=0$} with the total solution equation. We will assume at the initial state (t_0=0), the system is at rest {y(0)=0}, therefore:

$$y(0) = 0.02 + B = 0 \Rightarrow B = -0.02$$

Therefore y(t)= y_p(t)+ y_p(t)= $0.02e^{2t} - 0.02e^{-0.01t}$

Key points from this section are:

(1) Total solution consists of particular (with input) solution and homogeneous solutions (input set to 0). Homogeneous solution is considered natural response since x(t)=0.

(2) To find all the constants, equations based on system condition at a given time (t_0) are required. It is common to assume that system is at rest at time t=0 therefore y(0)=0.

(3) In the example and description presented in this section, we used first order equations. The same approach may be applied to higher order linear constant-coefficient differential equations such as the one shown by the following general form:

$$\sum_{j=0}^{P} a_j \frac{d^j y(t)}{dt^j} = \sum_{k=0}^{Q} b_k \frac{d^k x(t)}{dt^k}$$

- Linear Constant-Coefficient Difference Equations

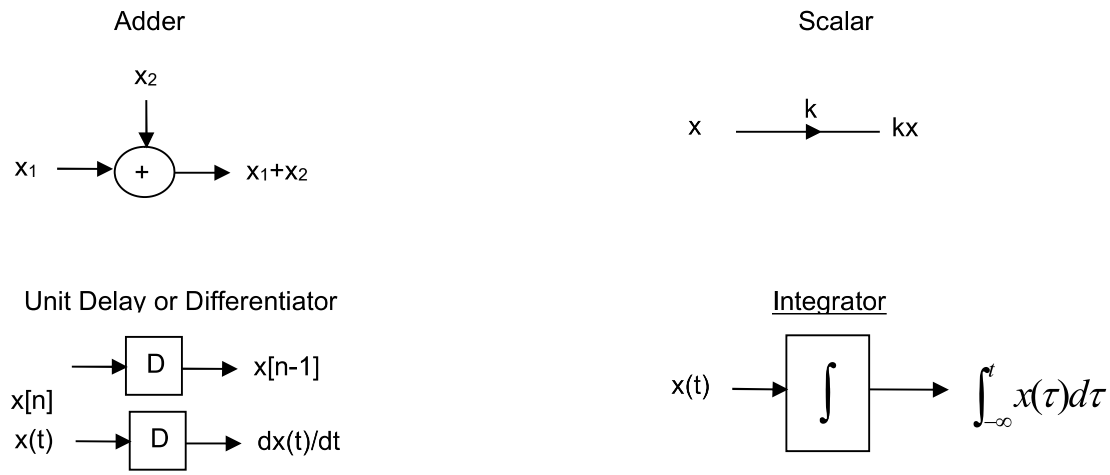
Linear constant-coefficient difference equations are the Discrete-Time version of the Continuous-Time Invariant Linear constant-coefficient differential equations. The general form of Linear constant-coefficient difference equations is shown below:

$$\sum_{j=0}^{P} a_j y[n-j] = \sum_{k=0}^{Q} b_k x[n-k]$$

The process to obtain the solution to this higher order difference equation is the same as the higher order differential equation in continuous time. The solution consists of particular solution (with input, $\sum_{j=0}^{P} a_j y[n-j] = \sum_{k=0}^{Q} b_k x[n-k]$) and homogeneous solutions (input is 0, $\sum_{j=0}^{P} a_j y[n-j] == 0$). The Homogeneous solution is considered natural response since input is set to 0.

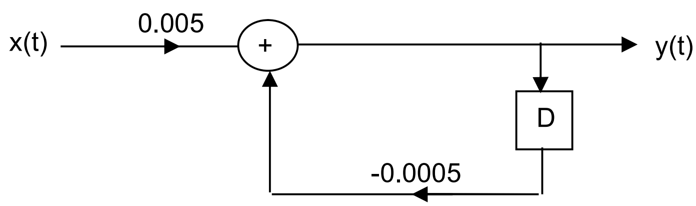
To find all the constants, equations based on system condition at given time is required. It is common to assume that n≤n_0 was at rest as an initial condition {$\sum_{j=0}^{P} a_j y[n_0] = 0$}.

- Graphical representation of the difference and differential equations
  One advantage of these equations is that they can be intuitively and easily described by graphical representation. Here are some of the common symbols used in the graphical representations:

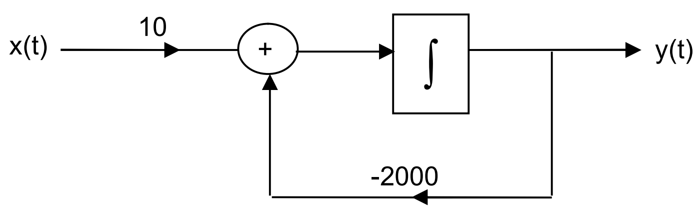


For example the differential equation $200y(t)+0.1\dfrac{dy(t)}{dt}=x(t)$ may be rewritten as:

$y(t)=-0.0005\dfrac{dy(t)}{dt}+0.005x(t)$ and can be represented graphically as shown below:



The same system may also be represented using integrators by rewriting it as
$y(t)=\int_{-\infty}^{t}[10x(\tau)-2000y(\tau)]d\tau$ and presented graphically as:



2.7. Chapter Summary

This section is a summary of key concepts from this chapter.

- Discrete-Time Convolution Sum

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

- Continuous-Time Convolution Integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

- Linear Time-Invariant (LTI) System Properties
  - Associativity
  - Causality
  - Commutativity
  - Distributivity
  - Invertibility
  - Linearity
  - Memory
  - Stability

2.8. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 2

- Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 2

- Stremler, F. Introduction to Communication Systems (1990) Addison-Wesley Publishing Company
  Chapt 2

- Birkhoff, G. Ordinary differential equations (1978) J. Wiley and sons

2.9. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 3. Fourier Series Representation of Periodic Signals

Key Concepts and Overview

- Overview & History of Fourier Series
- Complex Exponential Signals and LTI System Responses
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Continuous-Time Fourier Series
- Continuous-Time Fourier Series Properties
- Fourier Series Representation of Discrete-Time Periodic Signals
- Discrete-Time Fourier Series Properties
- Application of Fourier Series in LTI systems
- Filtering in Continuous-Time and Discrete-Time
- Additional Resources

3.1. Overview & History of Fourier series

This chapter introduces Fourier series in the context of Linear Time Invariant (LTI) Systems when input signal is periodic. The concepts will be developed both in Continuous-Time and Discrete-time. As discussed earlier, all periodic signals can be represented using the complex exponential form. Additionally, it will be shown that most periodic signals may be represented as a Fourier series which is a weighted sum of harmonics (integer multiples) of the fundamental frequency.

Fourier series and LTI system properties lead to relationships between input and output which are useful in finding the LTI system response to periodic signals. In future chapters, the concepts developed here will be broaden to include aperiodic (not periodic) signals.

From a historical point-of-view, the work started with Euler in 1748, Fourier made the key observation that periodic signals can be represented as a trigonometric series in 1802 and he also understood that this is a very important class of signals.

Dirichlet in 1829 developed the precise mathematical representation, now known as Fourier series. But it was not until 1965, when Cooley and Tukey discovered a method called Fast Fourier Transform (FFT) which enabled engineers to design computer programs that would perform functions such as filtering fast enough to make it useful in real time. FFT along with the advent of higher performance computers have led to use for Fourier transforms and series is design majority of signal processing devices.

In a nutshell, Fourier series allows us to represent any signal as a series of harmonic signal components. To filter any frequency, simply zero the coefficient for the corresponding harmonic. In general, we can reshape any signal by simply changing the coefficient of its harmonic components.

Although in this text our interest in Fourier series is in its application to audio, video, electrical and computing fields, the Fourier series concept has much wider uses including heat transfer and mechanical systems.

3.2. Complex Exponential Signals and LTI System Responses

As discussed earlier, Complex Exponential signals can be used to represent any periodic signal. Since we are focusing on LTI systems, the response to a periodic signal will be a linear combination of the complex exponential terms.

Let's start by defining a few terms to use in our development of Fourier series for complex exponential signals and LTI systems. The following table defines input and responses:

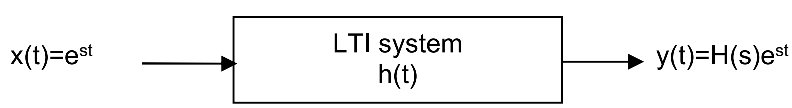
	Input	**Response**
Continuous-time	e^{st}	$H(s)e^{st}$
Discrete-time	z^n	$H(z)z^n$

Where:

- s and z are complex numbers but for the purposes here we can define them as:
  (1) s = jw "pure imaginary"
  (2) $z=e^{jw}$
- H(s) and H(z) are complex functions of s and z accordingly
- Response may or may not be complex
- Input e^{st} or z^n are referred to as the eigen function
- H(s) for a specific value of s_k is referred to as the eigen value associated with eigen function s_k. This is the value that is multiplied by eigen function to find the response.

Using the above definition for input, our next step is to find eigen values H(s) and H(z) in terms of the system impulse response h(t) and h[n] for Continuous-Time and Discrete-Time respectively.

❖ Determining the value of H(s) in terms of h(t) for a Continuous-Time LTI System
Here is a graphical restatement of the LTI system with impulse response h(t) and a periodic input signal x(t) and response y(t).



In order to find the eigen value H(s), we need to apply the convolution definition to find the response to this system:

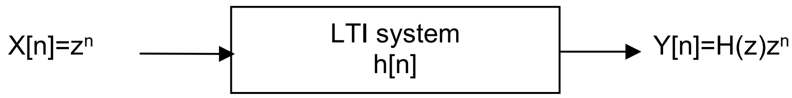
$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$y(t) = H(s)e^{st} \quad Where\, H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Notes:

- e^{jwt} is referred to as the eigen function (input)
- H(s) for a specific value of s_k is referred to as the eigen value associated with eigen function s_k. System Response is equal to eigen function multiplied by eigen function.

❖ Determining the value of H(z) in terms of h[n] for a Discrete-Time LTI System
Following a similar process as the Continuous-Time, H(z) can be determined in term of h[n].



$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]z^{(n-k)} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$y[n] = H(z)z^n \quad Where\, H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

Notes:

- z^n or e^{jwn} is referred to as the eigen function (input)
- H(s) for a specific value of z_k is referred to as the eigen value associated with eigen function z_k. System Response is equal to eigen function multiplied by eigen function.

❖ Response to Generalized Input

Generalized input signal may have many terms (eigen functions) which leads to system response being the sum of responses to each eigen function and its associated eigen value.

Starting with the Continuous time LTI system, Let's use the multi-term input

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

Applying the eigen function property, the response (output) to each input term can be written as follows:

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

Now, we can apply the superposition property since the system is linear which means response to the sum of inputs is the sum of individual responses:

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

We can generalize the above derivation for LTI systems as:

$$x(t) = \sum_k a_k e^{s_k t} \rightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

Similar process can be used to derive the generalized response for Discrete-Time LTI system which is shown below:

$$x[n] = \sum_k a_k z_k{}^n \rightarrow y(t) = \sum_k a_k H(z_k) z_k{}^n$$

❖ Examples - eigen value and eigen function

- Example 1: In a LTI System, output is related to input by a time shift of -4 find the eigen value associated with the eigen function $x(t) = e^{j2t}$

  <u>Solutions</u>
  $y(t) = x(t+4) = e^{j2(t+4)} = e^{j8}\, e^{j2t}$

  So we can say that eigen function is $e^{j2t} = e^{st} = e^{jwt}$ which means s=2j or w=2.
  Also eigen value $H(s) = e^{4s}$ for s=j2 is $H(s=j2) = e^{j8}$

  Note: $h(t) = \delta(t+4)$, so we can use convolution to confirm the output.

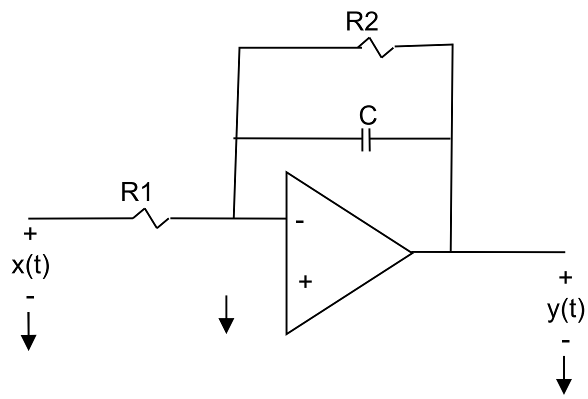
- Example 2: Given the eigen value $H(s) = e^{-2s}$, find the response to the input (eigen function) x(t) = cos5t

  <u>Solutions:</u>
  x(t) = cos(5t) = cos(wt) → w=5 rad/s → s=j5
  $y(t) = H(s)x(t) = e^{-2s}\cos(5t) = e^{-j10}\cos(5t)$
  Applying Euler's Identify $\cos(a) = ½\,(e^{ja} + e^{-ja})$ → $y(t)= ½\,(e^{j(5(t-2)} + e^{-j(5(t+2))})$

- Example 3: Given the eigen value $H(s) = e^{-2s}$, find the response to the input x(t) = cos(5t) + sin(9t).
  Note: Find eigen value for each component.

  <u>Solutions:</u>
  "*Student Exercise"*

- Example 3: Find eigen value, H(s) in-terms of R and C, for the following Circuit using ideal OpAmp Model):



<u>Solutions:</u>
"*Student Exercise"*

3.3. Fourier Series Representation of Continuous-Time Periodic Signals

As mentioned earlier, Fourier series is a representation of Continuous-Time periodic signals in terms of the weighted sum of a signal's harmonics. The first step in the Fourier Series development process is to define the periodic signal. A signal is periodic, if for some positive value of Period, T, the following equality holds:

$x(t) = x(t+T)$ for all t.

Fundamental period of x(t) is the minimum, positive, nonzero value of T that satisfies the above equation. Further, $W_0 = 2\pi/T$ is referred to as fundamental frequency.

Harmonically related periodic signal representations with a fundamental frequency of w_0 simply use signals with frequencies that are integer multiples of w_0. This means that the harmonic signals have period T_k which is a fraction of the fundamental period $T= 2\pi/w_0$. Below are some examples of harmonically related signals:

- Sinusoidal Signal, $x_1(t) = \cos(w_0 t)$
  The harmonically related signals of $x_1(t)$ can be represented as:

  $$\Phi_k(t) = \cos(k w_0 t) = \cos(k(2\pi/T)t) \; where\ k = 0, \pm 1, \pm 2, \ldots$$

- Pure Imaginary Exponential Signal, $x_2(t) = e^{jw_0 t}$
  The harmonically related signals of $x_2(t)$ can be represented as:

  $$\Phi_k(t) = e^{jkw_0 t} = e^{jk(2\pi/T)t} \; where\ k = 0, \pm 1, \pm 2, \ldots$$

The following facts are important to consider when working with the harmonically related periodic signal representations:

(1) Each harmonics has a frequency which is a multiple of the fundamental frequency.
(2) Signal component with k=0 is constant
(3) Signal component with k=+1 and -1 is referred to as the first harmonics or fundamental
(4) Signal component with k=+2 and -2 is referred to as the second harmonics
(5) Signal component with k=+N and -N is referred to as the Nth harmonics

The linear combination of harmonically related signals is also periodic which means that the following general form of harmonically related signals is periodic:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}}$$

Fourier Series Representation

This representation of a periodic signal is referred to as the "Fourier Series Representation" where a_k is referred to as the Fourier Coefficient.

- Examples - Fourier Series Representation

  - Example 1: Plot a periodic signal x(t), with fundamental frequency $w_0=\pi$ with the following Fourier series Coefficients:

    $a_0 = 1$; $a_1=a_{-1}=1/3$; $a_2=a_{-2}=1/2$; all other coefficients are 0

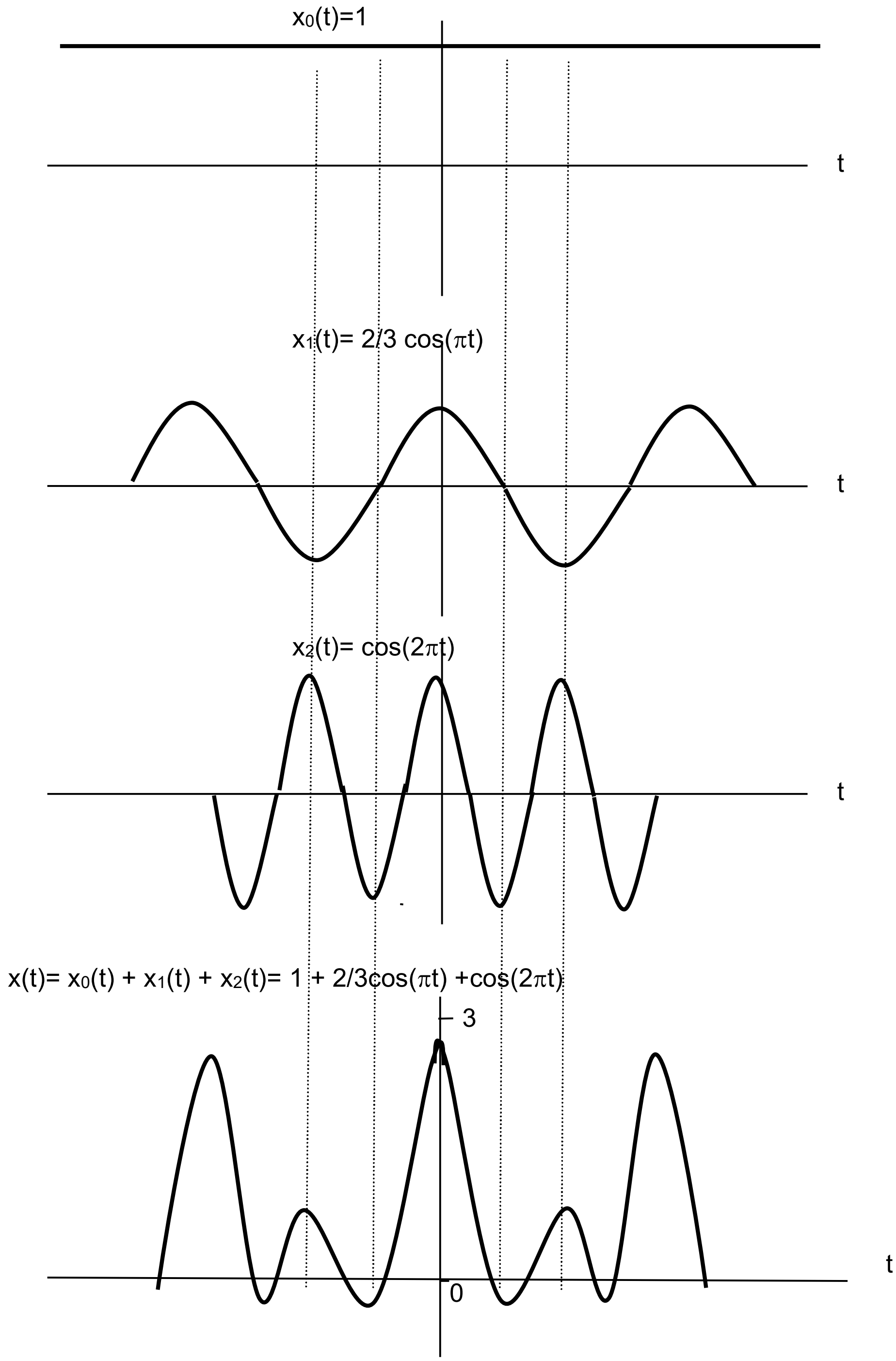
    <u>Solutions</u>

First write out the Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \frac{1}{2}e^{-j2\pi t} + \frac{1}{3}e^{-j\pi t} + 1e^0 + \frac{1}{3}e^{j\pi t} + \frac{1}{2}e^{j2\pi t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = 1 + \frac{1}{3}(e^{j\pi t} + e^{-j\pi t}) + \frac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})$$

Apply Euler's Identity → $(e^{ja} + e^{-ja}) = 2\cos(a)$
$x(t) = 1 + 2/3 \cos(\pi t) + \cos(2\pi t)$


$x_0(t)=1$
t
$x_1(t)= 2/3 \cos(\pi t)$
t
$x_2(t)= \cos(2\pi t)$
t
$x(t)= x_0(t) + x_1(t) + x_2(t)= 1 + 2/3\cos(\pi t) +\cos(2\pi t)$
3
t
0

- Example - Plot a periodic signal x(t), with fundamental frequency $w_0=\pi/2$ with the following Fourier series Coefficients:

  $a_0=1$; $a_1=a_{-1}=2$; $a_2=a_{-2}=3$; $a_3=a_{-3}=1/5$; all other coefficients are 0

  Verify your results using MATLAB.

  Solutions
  "Student Exercise"

- Example – Draw the time domain and frequency domain, and find values of the non-zero Fourier series coefficients (a_k) for the following signals:
  - a) A DC signal with magnitude of 2.
  - b) $x(t) = e^{j50\pi t}$
  - c) $x(t) = \sin(200\pi t)$

  Solutions
  "Student Exercise"

❖ Alternative Forms of Fourier Series Representations
There are three forms of Fourier Series Representation which are commonly used. All these forms are mathematically equivalent:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}}$$ *"This is the form that is most commonly use in this text"*

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(kw_0 t + \theta_k) \; where \; a_k = A_k e^{-j\theta_k}}$$

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{+\infty} \left[B_k \cos(kw_0 t) - jC_k \sin(kw_0 t)\right] where \; a_k = B_k + jC_k}$$

It is highly recommended that students proof that all three are mathematically equivalent.

❖ Calculation of the Fourier Series Representation of a Continuous-Time Periodic Signal
In order to derive the Fourier Series Representation, we need to have the original signal's fundamental frequency and Fourier Series Coefficient a_k. The fundamental frequency can be obtained from the signal. Below is the process for finding the Fourier Series Coefficient, a_k.

We know that any periodic signal can be represented by the following Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$ Fourier Series Synthesis Equation

Our objective here is to find the Fourier Coefficients, a_k for a given signal x(t). We will start by multiplying both side of the Fourier Series Synthesis Equation by e^{-jnwot}.

$$e^{-jnw_0 t} x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}$$

Next, integrate both side over one fundamental period (from 0 to $T=2\pi/w_0$)

$$\int_0^T e^{-jnw_0 t} x(t)dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t} dt$$

$$\int_0^T e^{-jnw_0 t} x(t)dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t} dt\right]$$

Side Bar

We can simplify the integral inside the bracket as shown below using Euler's identity:

$$\int_0^T e^{j(k-n)w_0 t}dt = \int_0^T \cos(k-n)w_0 t\ dt + j\int_0^T \sin(k-n)w_0 t\ dt$$

When (k-n)= 0 or k=n then $\int_0^T e^{j(k-n)w_0 t}dt = \int_0^T dt + j\int_0^T 0\ dt = T$

When (k-n)≠ 0 or k≠n then $\int_0^T e^{j(k-n)w_0 t}dt = \cos ine\ area\ for\ one\ period + \sin e\ area\ for\ one\ period$

$$= \ 0 \ + \ 0 \ = \ 0$$

Therefore $\sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0 t}dt\right] = a_k T$

Using the derivation of the side bar, we can rewrite the above equation as:

$$\int_0^T e^{-jkw_0 t}x(t)dt = a_k T$$

Which can be re-arranged to find the Fourier Coefficient a_k

$$a_k = \frac{1}{T}\int_0^T e^{-jkw_0 t}x(t)dt \quad or \quad a_k = \frac{1}{T}\int_T e^{-jkw_0 t}x(t)dt$$ Fourier Series Analysis Equation

- Summary of Fourier Series equations for a periodic Continuous-Time signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$ **Synthesis Equation**

$$a_k = \frac{1}{T}\int_T e^{-jkw_0 t}x(t)dt = \frac{1}{T}\int_T e^{-jk(2\pi/T)t}x(t)dt$$ **Analysis Equation**

* {a_k} is referred to as the Fourier Series Coefficient or the spectral coefficients of x(t)

Note of interest: Euler and Lagrange both knew this equation in 18th century but they did not see it as important since they did not feel it cover large enough class of periodic signal. Fourier a hundred years later showed that they indeed cover a large class which is the reason these equations are named for Fourier.

- Example - Fourier Series Application
  - Example - What are the Fourier Series Coefficients of x(t)=sin(w_ot)?

    Solutions

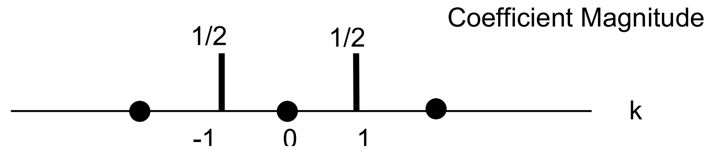
    You can use the analysis equations to find the coefficient $a_k = \frac{1}{T}\int_T e^{-jkw_0 t}x(t)dt, \quad w_o = 2\pi / T$

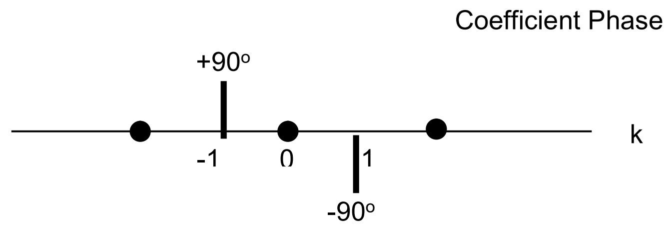
    But the easier way would be to use Euler's identity $\sin a = \frac{1}{2j}(e^{ja} - e^{-ja})$

$$x(t) = \sin\left(w_o t\right) = \frac{1}{2j} e^{jw_o t} - \frac{1}{2j} e^{-jw_o t}$$

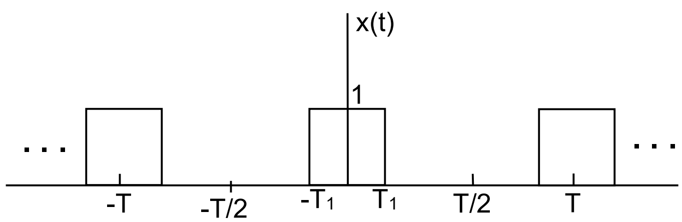
$a_1 = 1/2j$, $a_{-1} = -1/2j$, $a_k = 0$ for $k \neq -1$ & $k \neq +1$

Below is the bar graph of the Fourier Series Coefficient:





➢ Example: What is the Fourier Series Coefficient for the period Square Wave where:



$x(t) = 1$ for $|t| < T_1$
$\quad\quad 0$ for $T_1 < |t| < T/2$

$$a_k = \frac{1}{T}\int_T e^{-jkw_0 t} x(t)dt = \frac{1}{T}\int_T e^{-jk(2\pi/T)t} x(t)dt$$

"Fourier Series Coefficient can be calculated over any period", for this problem period from –T/2 to T/2 is one of the easiest.

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t} dt$$

for k=0 → $a_o = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

for k≠0 → $a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t} dt = \frac{2}{kw_o T}\left[\frac{e^{jkw_0 T_1} - e^{-jkw_0 T_1}}{2j}\right] = \frac{2\sin(kw_o T_1)}{kw_o T} = \frac{\sin(kw_o T_1)}{k\pi}$

Note: the above simplification uses Euler's identity $\sin a = \frac{1}{2j}(e^{ja} - e^{-ja})$ & $w_o = 2\pi/T$

For a square wave with 50% duty cycle we have $T=4T_1$. Therefore we can rewrite the coefficients as:

for k=0 → $a_o = (2T_1)/(4T_1) = 1/2$

for k≠0 → $a_k = \frac{\sin(kw_oT_1)}{k\pi} = \frac{\sin(k(2\pi/4T_1)T_1)}{k\pi} = \frac{\sin(\pi k/2)}{k\pi}$

$a_1 = a_{-1} = 1/\pi$
$a_2 = 0$
$a_1 = a_{-1} = -1/3\pi$
…

In general we can conclude that
$a_k = 0$ for even values of k
$a_k = 1/k\pi$ with alternative signs for odd values of k

Therefore the signal can be represented as a linear combination of harmonically related exponentials:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} = \frac{1}{2} + \frac{1}{\pi}(e^{-jw_0t} - e^{+jw_0t}) - \frac{1}{3\pi}(e^{-j3w_0t} - e^{+j3w_0t}) + \ldots$$

or by applying Euler's identity, we get:

$$x(t) = \frac{1}{2} - \frac{2j}{\pi}\sin(w_ot) + \frac{2j}{3\pi}\sin(3w_ot) + \ldots$$

- Example – For the following signals draw the time-domain representation, find non-zero Fourier series coefficients (a_k) and draw the frequency domain representation:

a) DC signal with magnitude of 2.
b) $e^{j50\pi t}$
c) sin(200πt)

Solution:
student exercise.

3.4. Convergence of the Continuous-Time Fourier Series

Fourier Series can represent an extremely large class of periodic signals including square waves as shown earlier. The full representation is a linear combination of an infinite number of harmonically related complex exponentials. The immediate question is how many terms are sufficient in representing the signal accurately. From the previous section, we know that x(t) is fully described by the following:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

In a finite sum, the positive and negative infinity limits nees to be replaced by a finite value of (-N to N) which means $x_N(t)$ value is a N^{th} approximation of x(t) and is written as:

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$ N^{th} approximation of x(t)

To answer the question of when a finite sum is a good enough approximation, we need to calculate the approximation error as shown below:

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$ Approximation Error

The goodness of approximation is quantitatively measured by the energy of the error in one period as shown below:

$$E_N = \int_T \{Power\ of\ error\ in\ a\ period\} dt$$ Energy of Error over one period

or

$$E_N = \int_T |e_N(t)|^2 \, dt$$

An approximation with E_N equal to zero means that the original and approximated signal have the same energy so it is a good approximation to the original signal. It is important to realize that it does necessarily mean that the approximated signal is the same as the original signal x(t).

A signal has a Fourier Series representation if and only if:
as N→ ∞ , $e_N(t)$ → 0 and $x_N(t)$ → x(t)

These types of signals are said to converge which means as k goes to infinity, $a_k = \frac{1}{T}\int_T e^{-jkw_0 t} x(t) dt$

and $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$ converge.

The good news is that there are large classes of signals that have no convergence problem. And for these signals Fourier Series representation exists. Even though a Fourier Series representation exists for convergent signals, it does not mean that the Fourier Series representation and original signal have the same value at every point. For example, in the case of signals with discontinuities such as a square wave, at the discontinuities the Fourier Series Representation takes on the average value of the signal from the two sides of the discontinuity.

❖ Convergence conditions and Dirichlet's three conditions
Again, a signal does not converge if:

(1) $a_k = \frac{1}{T}\int_T e^{-jkw_0 t} x(t)dt$ does not converge (meaning a_k goes to ∞)

or

(2) At times a_k does converge but the sum $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$ does not converge

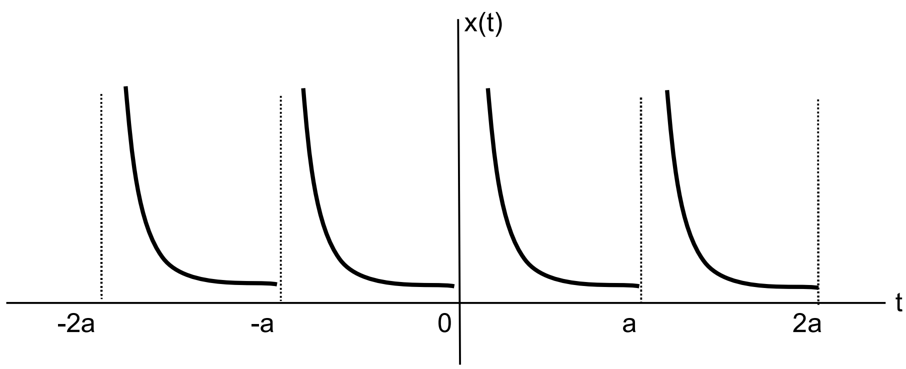
(meaning x(t) goes to ∞)

Criteria for deciding if a signal has a (Converging or valid) Fourier Series Representation
The periods that have finite energy over a single period will have Fourier Series Representation.

Dirichlet developed three conditions that when met, it is given that x(t) equals its Fourier Series representation excluding for values of t that x(t) is discontinuous.

- Condition 1
  Over any one period, x(t) Must be absolutely integrable; meaning $\int_T |x(t)|\, dt < \infty$

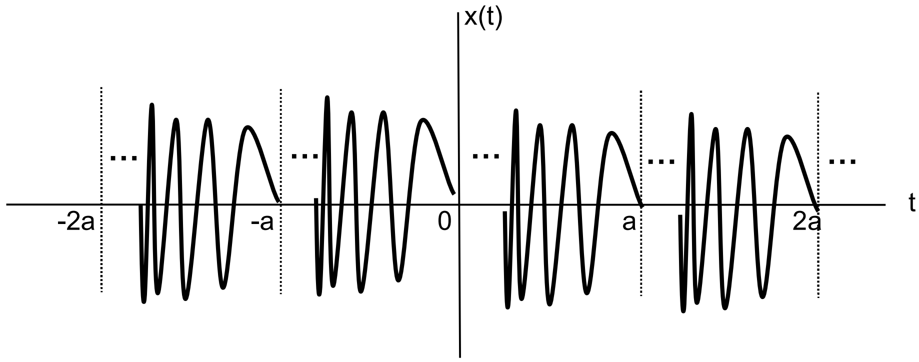
  For example the following x(t)t fails condition 1:



Other signals that fail this condition include $e^{1/\sin(t)}$ and tan(1/sin(t)).

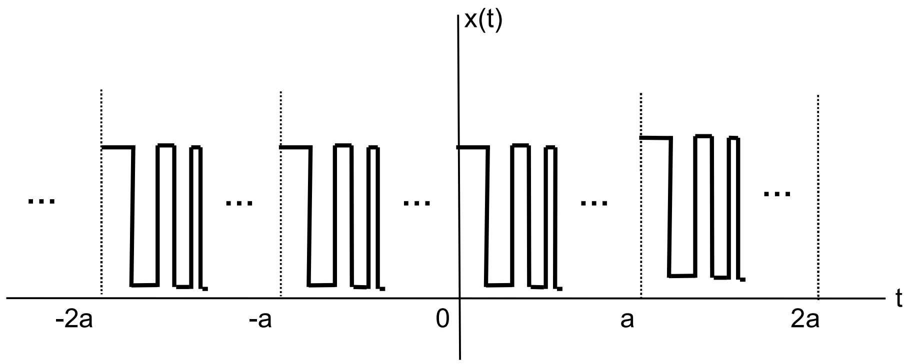
- Condition 2
  In any finite interval of time, x(t) has finite variation; meaning there is no more than a finite number or maximum of minimum during a single period.

  For example the following x(t) fails condition 2:



sin(1/sin(t)) is example of a signal that fail this condition.

- Condition 3
  In any finite interval of time, there are only a finite number of discontinuities. Each discontinuity also must be finite.



As you can see signal that do not meet the Dirichlet's three conditions are not common. It is safe to say that most natural signals are convergent and have a Fourier Series Representation.

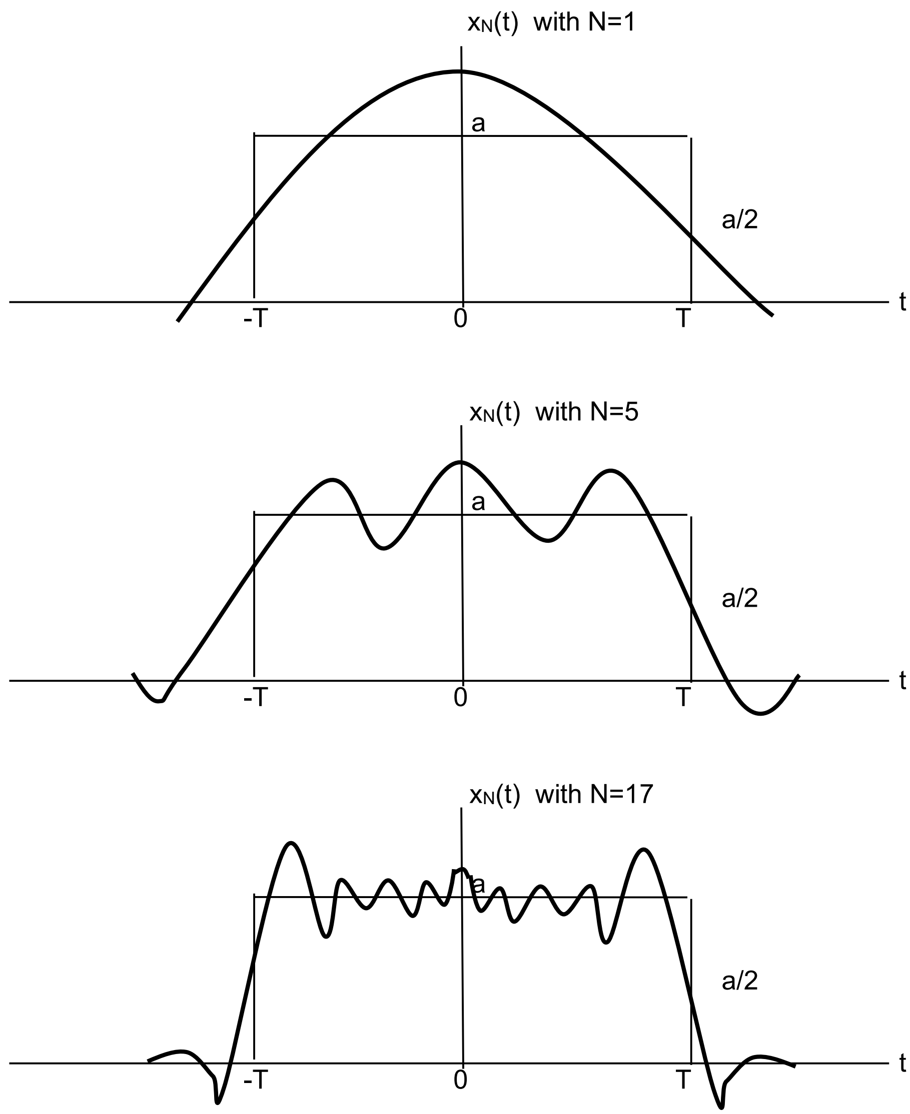
- Observations Relating to Signal Discontinuity
  For a periodic signal that has no discontinuity, the Fourier Series Representation converges and equals the original signal at every value of t as N→∞ where N is number of terms used in approximation.

  For a periodic signal with a finite number discontinuities in each period, the following holds true:

  - the Fourier Series Representation equals the signal everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the signal on each side of the discontinuity.

  - Gibbs Phenomenon is also present, meaning as the finite N (number of terms) increases, the peak amplitude of the error moves toward the discontinuities but the error peak values remain constant.

The following diagram demonstrates the two stated properties for a square wave as the number of terms to approximate the signal, N, is increased: (N=1, 5, 17)



3.5. Continuous-Time Fourier Series Properties

In this section we will cover some of the properties of Fourier Series which are used in future exercises and analysis of signals and systems. These properties not only provide us with insight, they are also crucial in reducing complexity.

The following notation is usedto show that a_k is the Fourier series coefficient of signal x(t):

$$x(t) \xrightarrow{FS} a_k$$

The proofs for the properties presented in this section are left to the reader. The proofs consist of plugging in the values into the synthesis or analysis equations in order to show that equality holds. The remainder of this section is dedicated to Fourier Series Properties:

- Linearity

  Let x(t) and y(t) denote two periodic signals with period T and fundamental frequency $w_o = 2\pi/T$. x(t) and y(t) have Fourier Series coefficient a_k and b_k respectively:

  $$x(t) \xrightarrow{FS} a_k$$
  $$y(t) \xrightarrow{FS} b_k$$

  Linearity property states that:

  $$z(t) = Ax(t) + By(t) \xrightarrow{FS} c_k = Aa_k + Bb_k$$

  c_k is Fourier coefficient of z(t) where A and B are constants.

- Time Shifting

  When a time shift is applied to a periodic signal x(t), the period T, fundamental frequency $w_o = 2\pi/T$ and magnitude of the signal is preserved.

  Time Shifting property states that:

  $$x(t) \xrightarrow{FS} a_k$$
  $$x(t - t_0) \xrightarrow{FS} e^{-jkw_o t_o} a_k$$

- Time Reversal

  When a time reversal is applied to a periodic signal x(t), the period T, fundamental frequency $w_o = 2\pi/T$ and magnitude of the signal is preserved.

  Time Reversal property states that:

  $$x(t) \xrightarrow{FS} a_k$$
  $$x(-t) \xrightarrow{FS} a_{-k}$$

  Two notes of interest relating to time reversal:

  - If x(t) is even, meaning x(t)=x(-t) → $a_k = a_{-k}$
  - if x(t) is odd , meaning x(t) = -x(-t) → $a_k = - a_{-k}$

  These two observations can be used to determine if a function is odd or even based on their Fourier

Series Coefficients.

❖ Time Scaling
Time scaling is an operation that may change the period of the signal. If x(t) is periodic with period T and fundamental frequency $w_o = 2\pi/T$ then the time scaled signal, x(bt), has the period T/b and fundamental frequency is bw_o. It is assumed that b is positive and real., is periodic with

Time Scaling property states that:

$$x(t) \xrightarrow{FS} a_k$$
$$x(bt) \xrightarrow{FS} \sum_{k=-\infty}^{\infty} a_k e^{jk(bw_o)t}$$

Note: Fourier series coefficient a_k has not changed, but the Fourier series has changed due to change in the fundamental frequency.

❖ Multiplication
Multiplication property states that:

$$x(t) \xrightarrow{FS} a_k$$
$$y(t) \xrightarrow{FS} b_k$$
$$Then$$
$$x(t)y(t) \xrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

❖ Conjugation and Conjugate Symmetry
Complex Conjugate is shown by * and means that j → -j so (a+jb)* = (a-jb). The Conjugate Symmetry states that:

$$x(t) \xrightarrow{FS} a_k$$
$$x^*(t) \xrightarrow{FS} a_{-k}{}^*$$

A number of useful observations to consider:

- If x(t) is real then $a_{-k} = a_k^*$ which means
  - ♦ a_0 is real since $a_0 = a_0^*$
  - ♦ $|a_{-k}| = |a_k|$ same magnitudes

- If x(t) is real and even using the fact that { $a_k = a_{-k}$ for even function} we can prove that a_k is real and even.
- If x(t) is real and odd using the fact that { $a_k = -a_{-k}$ for odd function}. then we can assert that a_k is pure imaginary and odd. However, a_0 must be 0 in this case.

❖ Parseval's Relation
Parseval relates average power of a signal to its Fourier Series Coefficient as shown below:

$$Average\ Power = \frac{1}{T}\int_T |x(t)|^2\ dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

To prove, replace x(t) with the synthesis equation and simplify the equation. The process is similar to proving the properties.

❖ Properties of Continuous-Time Fourier Series

Properties	**Periodic Signal** **x(t) and y(t) with $w_0 = 2\pi/T$**	**Fourier Series Coefficient** **a_k and b_k**
Conjugate symmetry	Real x(t)	$a_k = a_{-k}^*$
Conjugation	$x^*(t)$	a_{-k}^*
Convolution, Periodic	$\int_T x(\tau)y(t-\tau)d\tau$	$T\, a_k\, b_k$
Differentiation	$\frac{dx(t)}{dt}$	$jkw_0 a_k$
Frequency Shifting	$e^{jQw_o t}\, x(t)$	a_{k-Q}
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$ periodic & finite only if $a_0 = 0$	a_k / jkw_0
Linearity	Ax(t) + By(t)	$Aa_k + Bb_k$
Multiplication	x(t)y(t)	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Parseval's Relation Average Power for Periodic signals	$Average\ Power = \frac{1}{T}\int_T \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$	
Real and Even Signals	Real and even x(t)	a_k is real and even
Real and Odd Signals	Real and odd x(t)	a_k is purely imaginary and odd
Time Reversal	x(-t)	a_{-k}
Time Scaling	x(bt) b>0 → Period =T/b and $w_0 = 2\pi b/T$	a_k
Time Shifting	$x(t-t_0)$	$e^{-jkw_o t_o} a_k$

❖ Examples – Properties of Continuous-Time Fourier Series

➢ Example - Use the following properties of x(t) to determine the x(t) function:

1) x(t) is periodic with T=2
2) x(t) has Fourier Series Coefficient a_k
3) a_k =0 for |k| ≥ 2
4) x(t) is a real signal
5) DC component of the signal is 1/3
6) Magnitude of fundamental frequency component is 1/5.

Solution

Property 1 → $w_0 = 2\pi/T = \pi$
Property 2 & 3 → all a_k=0 except a_{-1}, a_0, a_1
Property 4 → $a_1 = a_{-1}^*$
Property 5 → a_0 = 1/3
Property 6 → a_1 = 1/5
Using prior information and synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = a_{-1}e^{jkw_0 t} + a_0 + a_1 e^{jkw_0 t} = (1/5)e^{-j\pi t} + 1/3 + (1/5)e^{j\pi t}$$

➢ Example – Find x(t+5) given x(t) has the following non-zero Fourier Series Coefficients:
a_{-1} = 2, a_0 = 5+j, a_4 = 3.

Solution

➢ Example – Given real function x(t) with non-zero Fourier Series Coefficients a_{-1}=a_1=3 and a_{-2}=a2=4,
a) Find the Fourier series Coefficients for x(9t).
b) What's fundamental frequency of x(9t).
c) What's the approximation of x(t) using Fourier Series.

Solution
a) Same as x(t)
b) $9w_0$
c) $$x(9t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk9w_0 t} = 4e^{j(-2)9w_0 t} + 3e^{j(-1)9w_0 t} + 3e^{j(1)9w_0 t} + 4e^{j(2)9w_0 t}$$
$$= 4e^{-j18w_0 t} + 3e^{-j9w_0 t} + 3e^{j9w_0 t} + 4e^{j18w_0 t}$$

- Example – Given that function x(t) has the Fourier Series Coefficients $\{a_{-1} = 2+j\ ,\ a_0 = 5,\ a_{+1} = 2-j\}$, determine if x(t) is real and/or even/odd.

  <u>Solution</u>

  $a_k = a_{-k}^*$ → x(t) is real
  It's neither odd or even.

- Example – For a 100 Hz square wave signal with 50% duty cycle, maximum of 5v and minimum of 0v,

  a) Find the Fourier Series Coefficients
  b) Find N-term approximation of x(t)

  <u>Solution</u>

3.6. Fourier Series Representation of Discrete-Time Periodic Signals

Development of Fourier Series Representation of Discrete-Time Periodic Signals is similar to the process used for Continuous-Time. The one key difference is that Discrete-Time Fourier Series representation is a finite series as opposed to infinite series representation required for Continuous-Time periodic signals. This difference means that the Discrete-Time Fourier Series does NOT have any convergence issue.

Now let's start the process of finding the relationship between the Discrete-Time signal and its Fourier Series (Synthesis and Analysis equations)

Again, periodic Discrete-Time Signal x[n] is a signal that meets the following definition:

- x[n]=x[n+N]
- The Fundamental period is the smallest positive integer N for which x[n]=x[n+N]
- Fundamental Frequency $w_0=2\pi/N$

- Discrete-Time Fourier Series Equations
  Any periodic signal can be represented mathematically in complex exponential form. Each term of the complex exponential signal may be represented by:

$$\phi_k[n] = e^{jkw_0 n} = e^{jk(2\pi/N)w_0 n} \quad k = \pm 1, \pm 2, \pm 3, ...$$

Signals represented by these terms have frequencies which are integer multiples of the fundamental frequency ($w_0=2\pi/N$). These terms represent the signal harmonics. We can restate the definition of a periodic signal in Discrete-Time by the fact that all terms of the complex exponential signal with the same relative location in the period are equal. The following equation restates this fact:

$\phi_k[n] = \phi_{(k+rN)}[n]$ where r is an integer and N is the fundamental Period.

The Discrete-Time Fourier Series is also a linear combination of harmonically related complex exponentials. The proof is similar to the Continuous-Time and starts by writing the convolution sum. The following equation pair are the general Discrete-Time Fourier Series Synthesis and Analysis Equations:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_o n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \quad \text{for } k = m, m+1, ..., m+N-1$$

Fourier Series Synthesis Equation

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_o n} = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

Fourier Series Analysis Equation

Where:
1) x[n] is a periodic signal
2) N is the Fundamental Period
3) m is an arbitrary integer
4) a_k is Fourier Series coefficient ($a_k = a_{\{k+N\}}$)

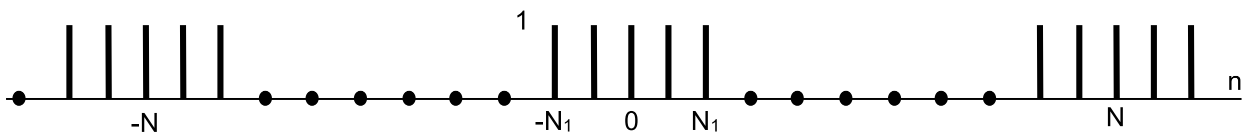
- Example - Fourier Series Representation of Discrete-Time Periodic Signals

  - Example1: Find the Fourier Series Coefficients for the signal x[n] = sin(w_0n).

    Solution
    Although we could use the analysis equation to find a_k, an easier way would be to use the Euler equation to reshape the signal into its linear exponential format select and the coefficients that would be equivalent to the x(t) representation with the synthesis equation.

    $$x[n] = \sin(w_0 n) = \frac{1}{2j}(e^{jw_0 n} - e^{-jw_0 n}) \rightarrow a_{1-} = -\frac{1}{2j};\ a_1 = \frac{1}{2j};\ otherwise\ a_k = 0$$

  - Example 2: Find the Fourier Series coefficient for the following discrete time periodic square wave signal.



    We see that the signal period is N. If we look at period -N/2 ≤ n ≤N/2 then only values in between $-N_1$ and $+N_1$ are one and the rest are zero. Thus we can write the analysis equation as:

    $$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n]e^{-jk(2\pi/N)n}$$

    By lettering n=m-N1, we can rewrite the equation as:

    $$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{e^{jk(2\pi/N)N_1}}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

    Applying the finite sum → $\sum_{k=0}^{M} b^k = \frac{1-b^{M+1}}{1-b}$ which allows us to write a_k as

    $$a_k = \frac{e^{jk(2\pi/N)N_1}}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m} = \frac{e^{jk(2\pi/N)N_1}}{N}\left[\frac{1-e^{-jk(2\pi/N)(2N_1+1)}}{1-e^{-jk(2\pi/N)}}\right]$$

- Question of Convergence for Discrete-Time Periodic Signal Fourier Series
  As discussed earlier, all Discrete-Time signal Fourier Series Coefficients converge so there is no issue with convergence unlike Continuous time.

  The Gibbs Phenomenon at discontinuity states that as the number of terms used in synthesis equation increasesthe error ripples become compressed but the error peak values remain constant. This holds true for Discrete-Time as well as Continuous-Time.

  For a signal, x[n], with the period N, x[n] is approximated using the following equation:

  Approximation of x(n) = $\hat{x}(n) = \sum_{k=-M}^{M} a_k e^{-jkw_0 n}$ Using terms from –M to M where:

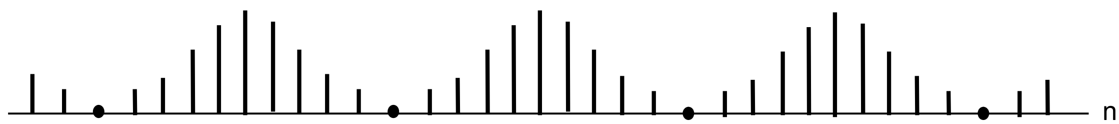
Even N → $-M \leq N/2$ *"x(t) approximation is error free when M=N/2"*

Odd N → $M \leq (N-1)/2$ *"x(t) approximation is error free when M=(N-1)/2"*

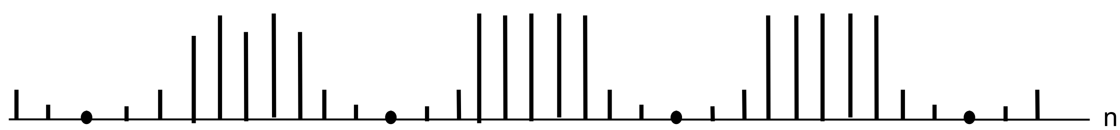
. *Note: N is the period.*

The following diagram demonstrates the effect of M on approximate square wave signals x[n]:

Approximation of x[n] with M=1

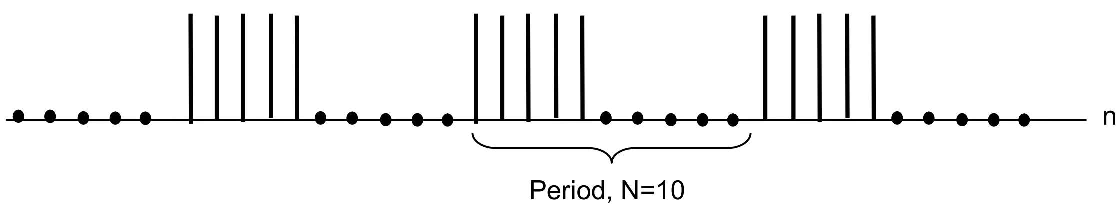


Approximation of x[n] with M=3



Approximation of x[n] with M=5

"Approximation is error free since M=N/2=5"



3.7. Discrete-Time Fourier Series Properties

The properties of Discrete-Time Fourier Series and Continuous-Time Fourier Series are similar. Below is a summary table of Discrete-Time Fourier Series Properties:

Properties	**Periodic Signal** **x[n] and y[n] with $w_0 = 2\pi/N$**	**Fourier Series Coefficient** **a_k and b_k "Period N"**
Conjugate symmetry	Real x[n]	$a_k=a_{-k}^*$
Conjugation	x*[n]	a_{-k}^*
Convolution, Periodic	$\sum_{l=<N>} x[l]y[n-l]$	N a_k b_k
First Difference	X[n] - X[n-1]	$(1-e^{-jkw_0})a_k$
Frequency Shifting	$e^{jQw_o n}\ x[n]$	a_{k-Q}
Linearity	Ax[n] + By[n]	$Aa_k + Bb_k$
Multiplication	X[n]y[n]	$\sum_{i=<N>} a_i b_{k-i}$
Parseval's Relation Average Power for Periodic signals	$Average\ Power = \frac{1}{N}\sum_{n=<N>} \lvert x[n] \rvert^2 = \sum_{k=<N>} \lvert a_k \rvert^2$	
Real and Even Signals	Real and even x[n]	a_k is real and even
Real and Odd Signals	Real and odd x[n]	a_k is purely imaginary and odd
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ periodic & finite only if a_0 =0	$(\frac{1}{1-e^{-jkw_0}})a_k$
Time Reversal	x[-n]	a_{-k}
Time Scaling	$x_{(b)}[n]$ = x[n/b] if n is a multiple of b 0 if n is not a multiple of b "Periodic with Period bN"	(1/b)a_k "Periodic with Period bN"
Time Shifting	X[n - n_0]	$e^{-jkw_o n_o} a_k$

- Examples – Discrete-Time Fourier Series Properties

  - Example – Write the expression for a Discrete-Time signal with the following characterization:

    1) Real periodic signal with period of 12
    2) With nonzero Fourier Series Coefficients:
    $a_0=-2$, $a_{-1}=-5j$, $a_6= a_7= a_8=3$, $a_{-14}= a_{-15}=-(2+j)$.

    Solution:
    use the properties:
    . Real → $a_k= a_{-k}^*$
    Discrete-Time Periodic → $a_k= a_{k+12}$
    $w_0 = 2\pi/N = 2\pi/12 = \pi/6$ rad/sec
    resulting in:
    $a_{-6}=3$, $a_{-5}=3$, $a_{-4}=3$, $a_{-3}=2+j$, $a_{-2}=32+j$, $a_{-1}=5j$, $a_0=-2$, $a_1=-5j$, $a_2=2-j$, $a_3=2-j$, $a_4=3$, $a_5=3$.

$$x[n] = \sum_{k=<N>} a_k e^{jkw_o n} = \sum_{k=-6}^{5} a_k e^{jkn\pi/6}$$

  - Example – Given two signals with period of 10 and the following nonzero Fourier Series Coefficients :
    x[n] → $a_0=5$, $a_1=29$, $a_3=-30$.
    y[n] → $b_1=3$, $b_5=10$, $b_6=-15$.

    Calculate c_k Fourier Series Coefficients of X[n]Y[n].

    Solution:

$$c_k = \sum_{l=<N>} a_l b_{k-i}$$

3.8. Application of Fourier Series in LTI systems

Fourier Series represents signals as linear combinations of weighted harmonics and LTI systems are linear by definition. These two facts enable us to determine the response to the LTI system by applying the concept of Fourier Series.

In this section we will first use the Fourier Series to calculate the system response in Continuous-Time and then in Discrete-Time.

❖ Continuous-Time LTI System Response
Earlier in this chapter, the system function, H(s), was derived in terms of the system impulse response of LTI system , h(t), which is show below:

$$System\ Function = H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

In general "s" is a complex number, but here we will use s as pure imaginary (s=jw) which results in:

$$Frequency\ Response = H(jw) = \int_{-\infty}^{\infty} h(\tau)e^{-jw\tau}d\tau$$

Input signal may be written as a Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}$$

Using the response equation $y(t) = H(s)e^{st,}$ and the above equation for input signal, LTI system response can be re-written as a Fourier series also:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0)e^{jkw_0t}$$

A few observations to consider include:

- Fourier Series Coefficient of response y(t) is $a_kH(jkw_0)$ where a_k is Fourier Coefficient of input and $H(jkw_0)$ is the frequency response.
- y(t) has the same Fundamental Frequency w_0 as the input.
- Fundamental Frequency $W_0=2\pi/T$

❖ Discrete-Time LTI System Response
Earlier in this chapter, the system function, H(Z), was derived in terms of system impulse response of LTI system , h[n], which is show below:

$$System\ Function = H(z) = \sum_{n=\infty}^{-\infty} h[n]z^{-n}$$

In general z is a complex number but here we will use |z|=1 → $z=e^{jw}$ which results in:

$$Frequency\ Response = H(e^{jw}) = \sum_{n=\infty}^{-\infty} h[n]e^{-jwn}$$

Input signal may be written as a Fourier Series Representation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

Using the response equation y[n] = H(z)z^n, and the above equation for input signal, the LTI system response can be re-written as a Fourier series also:

$$y[n] = \sum_{k=<N>}^{\infty} a_k H(e^{jkw_0}) e^{jkw_0 n}$$

A few observations to consider include:

- Fourier Series Coefficient of response y[n] is $a_k H(e^{jkw0})$ where a_k is Fourier Coefficient of input and $H(e^{jkw0})$ is the frequency response.
- y[n] has the same Fundamental Frequency w_0 as the input.
- Fundamental Frequency $W_0 = 2\pi/N$

❖ Examples – Application of Fourier Series in LTI Systems Properties

➢ Example – Signal, x(t) = 20 + 3 cos(3πt/8 – π/3), is passed through a low pass filter with cut off frequency of 10 Hz. Write the output signal equation.

Solution:

response, y(t) = 20 + 3 cos(3πt/8 – π/3)

➢ Example – x(t) is a square wave with a period of 1 msec. and 25% duty cycle with a maximum of 5V and a minimum of 0V. This signal is input to a band pass filter with cut off frequencies of 5,000 and 10,000 rad/sec. Calculate and plot the response in frequency and time domain.

Solution:

3.9. Chapter Summary

This section is a summary of key concepts from this chapter.

Continuous-Time Fourier Series

x(t) must be periodic and converging.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$ **Synthesis Equation**

$$a_k = \frac{1}{T}\int_T e^{-jkw_0 t} x(t)dt = \frac{1}{T}\int_T e^{-jk(2\pi/T)t} x(t)dt$$ **Analysis Equation**

Discrete-Time Fourier Series

X[n] must be periodic and is by definition converging.

$$x[n] = \sum_{k=<N>} a_k e^{jkw_o n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$ **Synthesis Equation**

$$for\ k = m, m+1, \ldots, m+N-1$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jkw_o n} = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$ **Analysis Equation**

Euler's Identity

$$e^{\pm ja} = \cos(a) \pm j\sin(a)$$

$$\cos(a) = \frac{e^{+ja} + e^{-ja}}{2}$$

$$\sin(a) = \frac{e^{+ja} - e^{-ja}}{2j}$$

3.10. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 3.

- Lathi, B. Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 3.

- Stremler, F. Introduction to Communication Systems (1990) Addison-Wesley Publishing Company
  Chapter 3.

- Nilsson, J. Electrical Circuits. (2004) Pearson.
  Chapter 16 and 17.

3.11. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 4. The Continuous-Time Fourier Transform

Key Concepts and Overview

- Introduction
- Fourier Transform of Aperiodic and Periodic Signals
- Fourier Transform Convergence
- Properties of the Continuous-Time Fourier Transform
- Systems Characterized by Linear Constant-Coefficient Differential Equations
- Additional Resources

4.1. Introduction

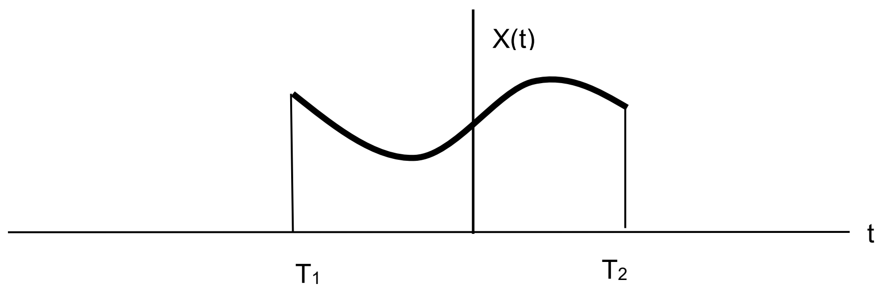
This chapter introduces Fourier Transform for Aperiodic signals. We will leverage the work from the pervious chapter on Fourier series for the periodic signal. Aperiodic signal will be approached as if it is a periodic signal with period, $T \rightarrow \infty$. This would also imply that the fundamental frequency, $W_0 \rightarrow 0$.

The resulting Fourier Transform has a number of useful properties that will be used to describe the characteristics of the original signal. Additionally, the Fourier Transform plays an important role in the study of signals & systems by providing a frequency spectrum analysis tool.

It will also be shown that Fourier Transform is applicable to periodic signals. This chapter is dedicated to Continuous-Time signal and Fourier Transform. Next chapter applies Fourier Transform to Discrete-Time aperiodic signals.

4.2. Fourier Transform for Aperiodic and Periodic Signals

The development of Fourier Transform for Continuous-Time Aperiodic signals starts by revisiting the definition of Aperiodic signal. An Aperiodic signal has non zero value of interested from some time value T_1 to T_2 which would fit within an interval t_0 to $(t_0 + T)$. As shown below:



We can consider that the signal shown above is a single period of a periodic signal with a period of T. With this consideration, the Continuous-Time Fourier Series synthesis and analysis equation can be used to write the following equations for x(t) approximation and a_k:

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \hat{x}(t) e^{-jkw_0 t}\, dt$$

for the Interval: $0 \le t \le T$ where $w_0 = 2\pi/T$

Note: $\hat{x}(t)$ *indicate estimate of x(t)*

Since the signal in reality is not periodic and the value of x(t) is zero outside of the interval, we can change the integration limit to infinity and use x(t) instead of approximation. These modifications results in the following analysis for calculation of Fourier Series Coefficient:

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jkw_0 t}\, dt$$

At this point we can define the Fourier Transform X(jw) as T multiplied by a_k.

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt}\, dt$$ Fourier Transform or Fourier Integral of x(t)

From the above equation, the Fourier Series Coefficient for a periodic signal may be calculated by setting w to kw_0 as shown below:

$$a_k = \frac{1}{T} X(jw)$$ where $w = kw_0$

using the above two equations, we can write the x(t) approximation in-term of X(jw) as shown below:

$$\hat{x}(t)=\sum_{k=-\infty}^{+\infty}\frac{1}{T}X(jkw_0)e^{jkw_0t}$$

we know that $w_o = 2\pi/T$ or $T = 2\pi/w_o$ so we can rewrite the above equation as:

$$\hat{x}(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}X(jkw_0)e^{jkw_0t}w_0=\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}X(jw)e^{jwt}dw$$

In the case of an Aperiodic signal, $T\rightarrow\infty$, $w_o\rightarrow 0$ and $dw \rightarrow 0$ which means the sum of terms in the above equation changes to an integral. This change results in the following integral which is referred to as an Inverse Fourier Transform:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jw)e^{jwt}dw \quad \text{Inverse Fourier Transform}$$

Below is a summary of the three equations resulting from the above derivations:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jw)e^{jwt}dw=F^{-1}\{X(jw)\} \quad \textit{Inverse Fourier transform equation}$$

$$X(jw)=\int_{-\infty}^{\infty}x(t)e^{-jwt}dt=F\{x(t)\} \quad \textit{Fourier transform or Fourier integral of } x(t)$$

$$a_k=\frac{1}{T}X(jw)=\frac{1}{T}X(jkw_o) \quad \textit{Fourier Series Coefficient if periodic}$$

- Example: Continuous-Time Fourier Transform for Aperiodic Signals

  - Example 1 – Find the Fourier Transform for the signal $x(t) = e^{-2t}u(t)$.

    Solutions:

$$X(jw)=\int_{-\infty}^{\infty}e^{-2t}u(t)e^{-jwt}dt=\int_{0}^{\infty}e^{-2t}e^{-jwt}dt=\int_{0}^{\infty}e^{-(2+jw)t}dt=-\frac{1}{2+jw}e^{-(2+jw)t}\Big|_0^{\infty}$$

$$X(jw)=\frac{1}{2+jw}$$

  - Example 2 – Find the x(t) with Fourier Transform

    X(jw) = 5   2 < w < 4
                 0   Otherwise

    Solution:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jw)e^{jwt}dw=\frac{1}{2\pi}\int_{2}^{4}5e^{jwt}dw=\frac{5}{j2\pi t}(e^{j4t}-e^{j2t})$$

- Example 3 - Find Fourier transform of the signal:

  $x(t) = e^{-a|t|} \{u(t-1) - u(t-5)\}$

  Solutions:
  Student Exercise.

- Example 4 - Find Fourier transform of the signal:

  $x(t) = e^{-a|t|} \{u(t+1) - u(t+5)\}$

  Solutions:
  Student Exercise.

- Example 4 - Draw x(t) and X(jw) when F{x(t)} = $\delta(w - 20\pi) - \delta(w + 20\pi)$

  Solutions:
  Student Exercise.

❖ *Fourier Transform of Periodic Signals*
The Fourier Transform of Continuous-Time Aperiodic signal derivation can be extended to find the Fourier Transform of periodic signals. Let's start with a Fourier Transform of the following form:

$X(jw) = 2\pi\delta(w-w_0)$

We can apply inverse Fourier Transform to find x(t):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(w-w_0)e^{jwt}dw = e^{jw_0t}$$

Note: $\delta(w-w_0)f(w)=f(w_0)$

We can generalize the above derivation to:

$x(t) = e^{+jkw_0t}$ which has a Fourier Transform of the form *$X(jw) = 2\pi\delta(w-kw_0)$*

As it was developed in the previous section, Periodic signals may be written as a linear sum of their harmonics using Fourier Series Coefficients which is shown below:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

Using the above relationships, the Fourier Transform of periodic signals may be written as:

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w-kw_0)$$

The Fourier Transform of a periodic signal with Fourier Series Coefficients a_K is an impulse train at the harmonic frequencies. The impulse train will be used in later chapters for the analysis of sampling systems.

Here is a summary of relationships for Fourier Transform of periodic signals:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} \quad \textit{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jkw_0t}dt \quad \textit{Fourier Series Analysis Equation}$$

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w-kw_0) \quad \textit{Fourier Transform Equation}$$

➢ Example 1– Find the Fourier Transform for the signal x(t) = sin(300t).

Solutions:

$w_0 = 300 = 2\pi/T \rightarrow T = 2\pi/300$

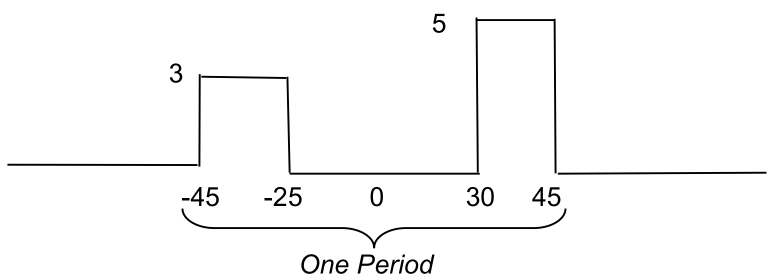
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jkw_0 t}\,dt = \frac{300}{2\pi}\int_{-\pi/300}^{\pi/300} (\sin 300t)e^{-jk300t}\,dt$$

$$a_k = \frac{300}{2\pi}\int_{-\pi/300}^{\pi/300} \frac{1}{2j}(e^{+j300t} - e^{-j300t})e^{-jk300t}\,dt$$

$$a_1 = -a_{-1} = \frac{1}{2j} \quad and \quad a_k = 0 \ for\ k \neq 0$$

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0) = \frac{\pi}{j}[\delta(w - w_0) - \delta(w + w_0)]$$

- Example 2 – Find the Fourier Transform of the following periodic signal with period of 90 mSec:



Solutions:
Student Exercise

- Example 2 – Find x(t) with Fourier Transform , F{x(t)} = sin(200πt)[u(t + 0.02) – u(-t + 0.02)].
Solutions:
Student Exercise

4.3. Fourier Transform Convergence

The discussion of Continuous-Time Fourier Transform for Aperiodic signals convergence is similar to the convergence of Fourier Series for periodic signals. Fourier Transform of x(t) converges and the equation $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$ accurately represents the original signal x(t) if only if x(t) has finite energy as shown by the following equation:

$$Energy = \int_{-\infty}^{\infty} |\, x(t)\,|^2 dt < \infty$$

If the original x(t) and x(t) approximated by inverse Fourier Transform have the same finite energy then the total error energy is 0 as shown below:

$$error = e(t) = x(t) - \hat{x}(t)$$

$$Energy = \int_{-\infty}^{\infty} |\, e(t)\,|^2 dt = 0$$

Note: $\hat{x}(t)$ *indicate estimate of x(t)*

Another way to state the same concept is that “the approximated signal has the same energy signature as the original x(t) signal”.

Much like the periodic case, Dirichlet conditions are sufficient to ensure the approximation of x(t) using Inverse Fourier Transform is equal to x(t) except at the discontinuities. As discussed earlier, the value at the discontinuities is equal to the average of the values on either side of the discontinuity.

Again, the three Dirichlet conditions required for convergence are listed here:

1) x(t) is absolutely integrable which means $\int_{-\infty}^{\infty} |\, x(t)\,| dt < \infty$.
2) x(t) has finite number of oscillations within any finite interval.
3) x(t) has finite number of discontinuities within any finite interval and each discontinuity is finite.

- Example – Fourier Transform Convergence

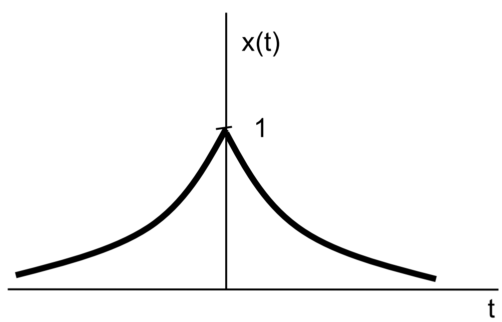
  - Example 1 – Determine the Fourier Transform For the following signal if it converges:

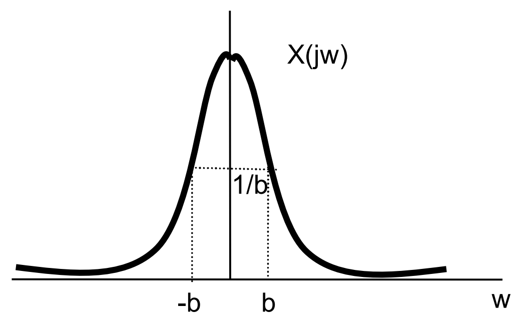
    $x(t)=e^{-b|t|}$ where b>0.

    Solution:
    Since all three of Drichlet's condition holds, X(jw) converges and x(t) has a valid Fourier series. The following process is used to calculate x(jw):

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{-\infty}^{0} e^{bt}e^{-jwt}dt + \int_{0}^{\infty} e^{-bt}e^{-jwt}dt = \frac{1}{b-jw} + \frac{1}{b+jw}$$

$$X(jw) = \frac{2b}{b^2 + w^2}$$





4.4. Properties of the Continuous-Time Fourier Transform

The development of Fourier transform X(jw) provides us with a frequency description of signal (frequency-domain) while x(t) provides us with time description of signal (time-domain). The following table presents the properties of Continuous-Time Fourier Transform:

Properties of Fourier Transform of Aperiodic Signal

Properties	**Aperiodic Signals: x(t) and y(t)**	**Fourier Transform: X(jw) and Y(jw)**
Conjugate symmetry	Real x(t)	X(jw)=X*(-jw)
Conjugation	x*(t)	$X^*(-jw)$
Convolution	x(t)*y(t)	X(jw)Y(jw)
Differentiation in Frequency	tx(t)	$j\frac{dX(jw)}{dw}$
Differentiation in Time	$\frac{dx(t)}{dt}$	jwX(jw)
Frequency Shifting	$e^{jw_o t}x(t)$	$X(j(w-w_0))$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{jw}X(jw)+\pi X(0)\delta(w)$
Linearity	Ax(t) + By(t)	AX(jw) + BY(jw)
Multiplication	x(t)y(t)	$\frac{1}{2\pi}X(jw)*Y(jw) \quad or$ $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(w-\theta))d\theta$
Symmetry	Real and even x(t) Real and odd x(t)	X(jw) is real and even X(jw) is purely imaginary and odd
Time & Frequency Scaling	x(bt)	$\frac{1}{\lvert b\rvert}X(\frac{jw}{b})$
Time Reversal	x(-t)	X(-jw)
Time Shifting	$x(t-t_0)$	$e^{-jwt_o}X(jw)$
Parseval's Relation Total Energy of the Aperiodic signal	$Total\ Energy = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(jw)\rvert^2\ dw$	

All of these properties are important in our analysis and to our ability to move between time-domain and frequency-domain view of the signal. Access to these two domains enables us to perform the analysis in the domain which provides the most effective environment for signal analysis. Later in this chapter, the Convolution and Multiplication property will be explored further due to their importance. We also will discuss Duality which is not listed in the table.

To prove the any of the relationships listed in the above Properties table, simply apply x(t) and X(jw) to either the Fourier Transform equation or Inverse Fourier Transform as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw = F^{-1}\{X(jw)\} \quad \textit{Inverse Fourier Transform equation}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = F\{x(t)\} \quad \textit{Fourier Transform or Fourier Integral of } x(t)$$

- Examples – Fourier Transform

  - Example 1 – Prove the frequency shifting Property of Fourier transform:

    $F\{e^{jw_o t}x(t)\} = X(j(w-w_0))$ or $e^{jw_o t}x(t) = F^{-1}\{X(j(w-w_0))\}$

    Solution:

  - Example 2 – Using the properties in the properties table find the Fourier transform in terms of F{x(t)}=X(jw) for the following signal:

    y(t) = 2 x(t - kπ/3) + 4 x(t + nπ/5)

    Solution:

  - Example 3 – Find Fourier Transform of y(t) = $\frac{dx(t)}{dt} + x(t-25)$ when Fourier Transform of x(t)is X(jw).

    Solution:

    $$Y(jw) = jwX(jw) + e^{-j25w}X(jw)$$

  - Example 4 – Find Fourier Transform of y(t) = $\frac{d(\sin(2000\pi t))}{dt} + \sin(200\pi(t-25))$.

    Solution:

❖ Convolution Property and Application

Convolution property states that a time-domain convolution is replaced by multiplication in frequency domain as shown below:

$$\boxed{y(t) = h(t) * x(t) \quad \overset{F}{\leftrightarrow} \quad F\{y(t)\} = Y(jw) = X(jw)H(jw)}$$

Here is the proof that Convolution Property holds true:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}\,dw$$

$$you\ can\ rewrite\ as\ a\ sum\ when\ w_0 \to 0\ \ and\ \ k \to \infty$$

$$x(t) = \lim_{w_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jkw_0)e^{jkw_0 t}w_0$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt \to H(jkw_0) = \int_{-\infty}^{\infty} h(t)e^{-jkw_0 t}\,dt$$

$$Applying\ LTI\ \ properties\ such\ as\ no\ cross-frequency\ \ products\ and\ \sup erposition$$

$$y(t) = \lim_{w_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)H(jw)e^{jwt}\,dw$$

$$\sin ce\ y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(jw)e^{jwt}\,dw$$

$$Then$$

$$Y(jw) = X(jw)H(jw)$$

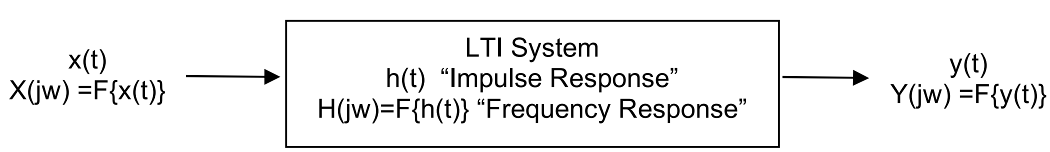
The importance of the Convolution property is the fact that it can be applied to LTI systems to determine the System Response. In earlier work, it is known that in time-domain the system response is equal to input x(t) convoluted with impulse response h(t):

y(t) = h(t) * x(t)

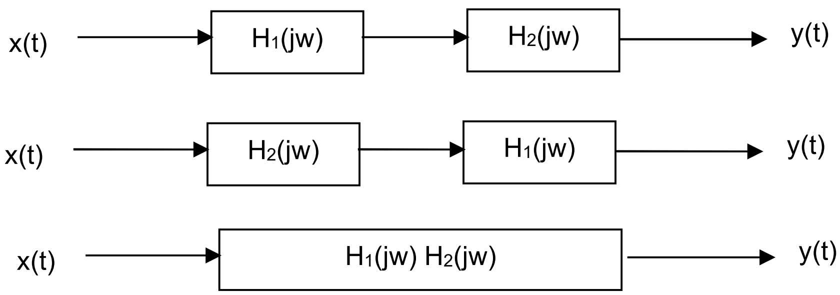
By taking the Fourier Transform of both side, we can state that in frequency-domain, response Y(jw) is the product of input X(jw) and Frequency Response H(jw):

Y(jw) = X(jw)H(jw) where X(jw) = F{x(t)}, Y(jw) = F{y(t)} and H(jw) = F{h(t)}.

Below is a system diagram annotated in time-domain and frequency-domain:



Like the Unit Impulse response, h(t), frequency response H(jw) completely characterizes the LTI System and it does not depend on the order. Therefore, the following system configurations are equivalent:



$Y(jw)=X(jw)H_1(jw)H_2(jw)$

Furthermore, H(jw) converges and exists if its energy is bounded. This means we can apply Drichlet's three conditions. If all three conditions are met then H(jw) converges and therefore it exists:

- x(t) is integrable; that is $\int_{-\infty}^{\infty} x(t)dt < \infty$
- x(t) has finite number of oscillations within any finite interval
- x(t) has finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

- Example - Given $y(t) = \int_{-\infty}^{t} x(t)dt$ is the response of an LTI system with input x(t) what is the impulse response h(t) of the LTI system?

  Solution:

  Take an Inverse Fourier Transform of $y(t) = \int_{-\infty}^{t} x(t)dt$

$$Y(jw) = \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

$$known: X(0)\delta(w) = X(jw)\delta(w)$$

$$Y(jw) = \frac{1}{jw}X(jw) + \pi X(jw)\delta(w)$$

$$Y(jw) = \{\frac{1}{jw} + \pi\delta(w)\}X(jw)$$

$$Known: Y(jw) = X(jw)H(jw)$$

$$Therefore: H(jw) = \frac{1}{jw} + \pi\delta(w)$$

Using FT property table → h(t) = u(t)

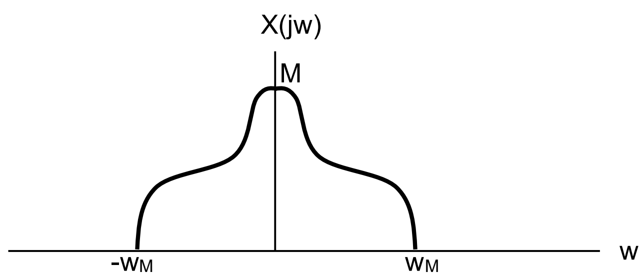
- Multiplication Property and Application
  Multiplication property states that a time-domain multiplication is replaced by convolution in frequency domain as shown below:

$$r(t) = x(t)p(t) \quad \overset{F}{\leftrightarrow} \quad F\{r(t)\} = R(jw) = \frac{1}{2\pi} x(jw) * H(jw) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta)P(j(w-\theta))d\theta$$

To prove this property take Fourier Transform of both sides of the time-domain equation and show that the right side is a convolution.

The multiplication property is used to scale signal. It is also used to move the signal up or down the frequency spectrum which is referred to as modulation.

- Example: x(t) is a signal with Fourier Transform X(jw).



Sketch the Frequency Spectrum of the signal resulting from the multiplication of x(t) and p(t)=cos(W_Mt).

Solution:
Student Exercise

Hints:
** Find the Fourier transform of p(t) and plot it.*
** Perform the convolution between the two signals. Remember the 1/2π factor.*

- Duality Property
  Fourier Transform and Inverse Fourier Transform exhibit symmetry such that by interchanging time and frequency variables in the transform, a new dual transform is derived. This is referred to as Duality.

For example the following is the time shift property of Fourier transform:

$$x(t-t_0) \quad \overset{F}{\leftrightarrow} \quad F\{x(t-t_0)\} = e^{-jwt_0} X(jw)$$

The dual pair of above transforms can be derived by exchanging time and frequency variables as shown below:

$$e^{-jw_0 t} x(t) \quad \overset{F}{\leftrightarrow} \quad F\{e^{-jw_0 t} x(t)\} = X(w-w_0)$$

❖ Time-Domain Signal, Fourier Transform and Fourier Series Coefficient
The following table summarizes some of the most common signals and their Fourier transforms. Also included is the Fourier Series Coefficient for the signal if the signal is periodic.

Time-Domain Signal, Fourier Transform and Fourier Series Coefficient

Signal, x(t) =F^{-1}{X(jw)}	**Fourier Transform, X(jw)=F{x(t)}**	[1] **Fourier Series Coefficient, a_k**
$\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(w - kw_0)$	$\frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jkw_0 t} dt$
$e^{jw_0 t}$	$2\pi\delta(w - w_0)$	a_1=1 a_k =0 for all other k values
cos(w_0t)	$\pi[\delta(w - w_0) + \delta(w + w_0)]$	a_1 = a_{-1} =1/2 a_k =0 for all other k values
sin(w_0t)	$\frac{\pi}{j}[\delta(w - w_0) - \delta(w + w_0)]$	a_1 = -a_{-1} =1/(j2) a_k =0 for all other k values
1	$2\pi\delta(w)$	a_0= 1 a_k =0 for all other k values
Periodic Square Wave x(t) = 1 |t| <T_1 0 T_1< |t| <T/2 with Period T.	$\sum_{k=-\infty}^{+\infty} \frac{2\sin(kw_0 T_1)}{k}\delta(w - kw_0)$	$\frac{\sin(kw_0 T_1)}{k\pi}$
Pulse Train in Time $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi k}{T})$	a_k = 1/T for all k
x(t) = 1 |t|<T_1 0 |t|≥T_1	$\frac{2\sin wT_1}{w}$	N.A.
$\frac{\sin Wt}{\pi t}$	X(jw) = 1 |w| < W 0 |w| > W	N.A.
$\delta(t)$	1	N.A.
u(t)	$\frac{1}{jw} + \pi\delta(w)$	N.A.
$\delta(t - t_0)$	e^{-jwt_0}	N.A.
$e^{-at}u(t)$ for *Re*{a}>0	$\frac{1}{a + jw}$	N.A.
$te^{-at}u(t)$ for *Re*{a}>0	$\frac{1}{(a + jw)^2}$	N.A.
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ for *Re*{a}>0	$\frac{1}{(a + jw)^n}$	N.A.

Note: [1]Fourier Series Coefficient exists only if signal is periodic.

4.5. Chapter Summary

This section is a summary of key concepts from this chapter.

- Continuous -time Fourier Transform
  x(t) may be aperiodic but must be converging.

  - Fourier Transform - Analysis Equation
    X(jw) is the Fourier Transform of x(t) and is aperiodic in frequency domain.

    $$X(jw) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

  - Fourier Inverse Transform - Synthesis Equation
    x(t) is the Fourier inverse Transform of x(t) and is aperiodic in time domain.

    $$x(t) = F^{-1}\{x(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}\,dw$$

  - Fourier Transform and Fourier Series relationship (valid only if x(t) is periodic)
    a_k =X(jw) = X(jkw_0)

4.6. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 4.

- Lathi, B. Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 3.

- Stremler, F. Introduction to Communication Systems (1990) Addison-Wesley Publishing Company
  Chapter 3.

- Nilsson, J. Electrical Circuits. (2004) Pearson.
  Chapter 16 and 17.

4.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 5. The Discrete-Time Fourier transform

Key Concepts and Overview

- Introduction
- Fourier Transform of Aperiodic and Periodic Signals
- Fourier Transform Convergence
- Properties of the Discrete-Time Fourier Transform
- Summary of Fourier Series and Transform Equations
- Systems Characterized by Linear Constant-Coefficient Differences Equations
- Additional Resources

5.1. Introduction

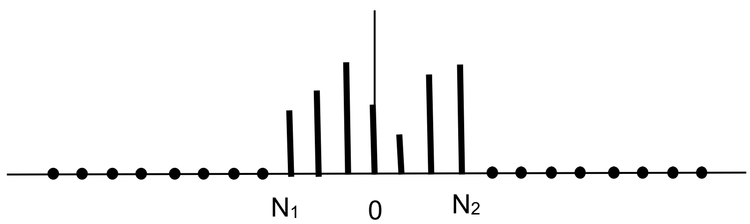
As discussed earlier, a major benefit of Fourier Transform is the ability to perform Frequency-Domain analysis for LTI systems in addition to time-domain analysis. Specifically, the convolution property in time-domain and its equivalent multiplication in frequency-domain relate the response and input of a LTI system.

Development of Discrete-Time Fourier Transform for Aperiodic signals follows a similar process to the Continuous-Time Fourier Transform development. As it was done in the last chapter, the development of Discrete-Time Fourier Transform starts with the Fourier series and the assumption that any aperiodic signal can be treated as a single period of a periodic signal.

Many of the Discrete-Time Fourier Transform properties are counterparts of the Continuous-Timer Fourier Transform properties. One difference is that the Discrete-Time Fourier Transform of an aperiodic signal is always periodic with period 2π.

5.2. Fourier Transform of Aperiodic and Periodic Signals

The development of Fourier Transform for Discrete-Time Aperiodic signals starts by revisiting the definition of an aperiodic signal. An Aperiodic signal has non zero values of interest from some time value N_1 to another time value N_2 which would fit within an interval n_0 to (n_0 + N). As shown below:



We can consider the signal above as a single period of a periodic signal with a period of N. With this consideration, the Continuous-Time Fourier Series synthesis and analysis equations can be used to write the following equations for x[n] approximation and a_k:

$$\hat{x}[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \hat{x}[n] e^{-jkw_0 n}$$

for the Interval: $0 \le n \le N$ where $w_0 = 2\pi/N$

The signal in reality is not periodic and the value of x[n] is zero outside of the finite interval, we can change the integration limit to infinity ($N \rightarrow \infty$ or $w_0 \rightarrow 0$) to fully represent the Aperiodic signal x[n]. By substituting the limits for the period and replacing the $\hat{x}[n]$ with x[n], the Fourier Series Coefficient equation can be written as:

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

Given $w=k(2\pi/N)=kw_0$, Discrete Fourier Transform is defined as:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$ Fourier Transform Equation (Analysis equation)

$X(e^{jw})$ is sometimes referred to as the spectrum of x[n].

Using the above definition, relationship between a_k and $X(e^{jw})$ may be represented by:

$$a_k = \frac{1}{N} X(e^{jkw_0})$$

Plugging the above value of a_k into the following Fourier Series equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

Results in:

$$x[n] = \frac{1}{N}\sum_{k=<N>} X(e^{jw})e^{jkw_0 n} \quad \text{where } N = 2\pi/w_0$$

$$x[n] = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jw})e^{jkw_0 n} w_o$$

As N increases and w_0 decreases, the above summation transitions to an integral where {dw=w_0 = (k+1)wo – kw_0} → 0.

$$x[n] = \frac{1}{2\pi}\int_N X(e^{jw})e^{jwn}dw$$

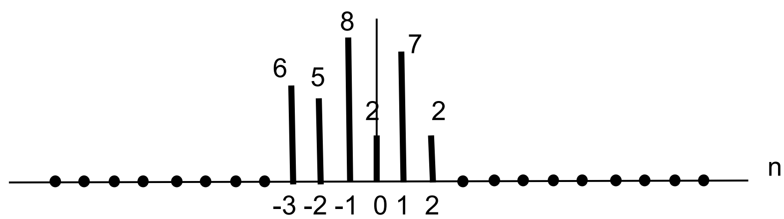
$X(e^{jw})e^{jwn}$ is periodic in frequency domain (w) with a period of 2π therefore x[n] can be written in terms of $X(e^{jw})$ as shown below:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw \quad \text{Inverse Fourier Transform (Synthesis equation)}$$

The above derivation of Discrete-Time Fourier Transform of Aperiodic signal can be summarized:

$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$	Fourier Transform Equation (Analysis equation)
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$	Inverse Fourier Transform (Synthesis equation)

- Example – Discrete-Time Fourier Transform for Aperiodic Signal
  - Example – Calculate Fourier Transform for the following x[n]:



Solution:

$$F\{x[n]\} = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-3}^{2} x[n]e^{-jwn} = 6e^{j3w} + 5e^{j2w} + 8e^{jw} + 2 + 7e^{-jw} + 2e^{-j2w}$$

- Example - Find and plot the magnitude and phase of Fourier Transform of $x[n]=a^n u[n]$ where $a<1$.

  Solution

  $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-jwn} = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n$$

  Applying the finite sum equation:

  $$X(e^{jw}) = \frac{1}{1-ae^{-jw}}$$

  Plot the magnitude and phase of $X(e^{jw})$ and use MATLAB to plot the equation.

- Example 2: Find and plot the magnitude and phase of Fourier Transform of

  $$x[n] = \begin{cases} 1 & \text{where } |n| \le N1 \\ 0 & \text{where } |n| > N1 \end{cases}$$

  Solution

  $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-N_1}^{N_1} e^{-jwn} = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

  let n=m-N1

  $$X(e^{jw}) = \sum_{m=0}^{2N_1} (e^{-jw})^{m-N_1} = e^{jwN_1} \sum_{m=0}^{2N_1} (e^{-jw})^m$$

  Applying Finite Sum Equation

  $$X(e^{jw}) = e^{jwN_1} \frac{1-(e^{-jw})^{2N_1+1}}{1-(e^{-jw})} = \frac{2\sin w(N_1+1/2)}{\sin(w/2)}$$

  Plot the magnitude and phase of $X(e^{jw})$ and use MATLAB to plot the equation.

❖ Fourier Transform of Discrete-Time Periodic Signals

The remainder of this section is focused on the derivation of Fourier Transform for Discrete-Time Periodic Signals. First, consider a periodic signal $x[n] = e^{jw_0 n}$ which has a period of 2π. We will suggest that the Fourier Transform of x[n] should be the sum of impulses at w_0, $w_0 \pm 2\pi$, $w_0 \pm 4\pi$, … . So here is the Fourier Transform of x[n]:

$$X[e^{jw}] = \sum_{l=-\infty}^{\infty} 2\pi\delta(w - w_0 - 2\pi l)$$

To check the validity of the Fourier Transform, one has to take the Inverse Fourier Transform of $X(e^{jw})$ and prove that x[n] is indeed a periodic signal:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(w - w_0 - 2\pi l)e^{jwn}dw = e^{jw_o n}$$

Note : there is only one impulse per 2π period.

The general form of a periodic signal (sequence) with period N and its transform can be written as follows:

$$\boxed{\begin{aligned} x[n] &= \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \\ X(e^{jw}) &= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(w - \frac{2\pi k}{N}) \end{aligned}}$$

Fourier Transform of a periodic signal can be directly constructed from the Fourier Series Coefficients as shown by the above equation.

- Example – Discrete-Time Fourier Transform of Periodic Signal
  - Example1: Find Fourier Transform of Periodic Signal: $x[n] = \cos w_0 n$ where $w_0 = \dfrac{2\pi}{5}$.

    <u>Solution</u>

    Apply Euler's identity:

    $$x[n] = \frac{1}{2} e^{jw_o n} + \frac{1}{2} e^{-jw_o n}$$

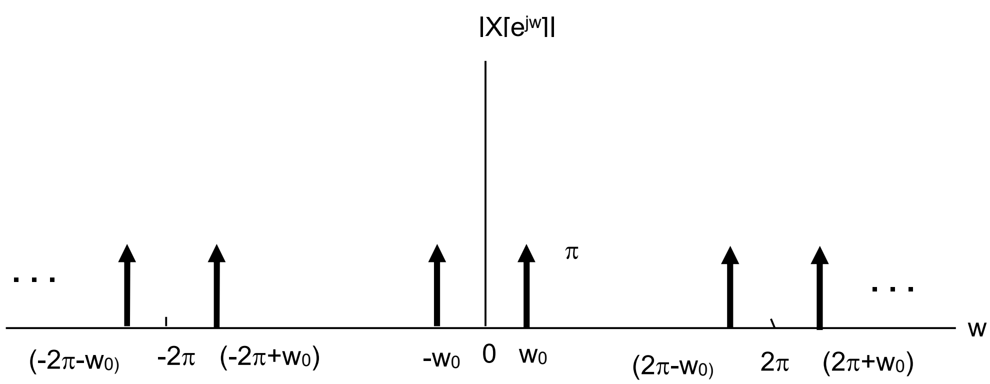
    Now we can apply the Fourier Transform equation where $w_0 = \dfrac{2\pi}{5}$ → N=5

    $$X[e^{jw}] = \sum_{l=-\infty}^{\infty} 2\pi\delta(w - w_0 - 2\pi l) = \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(w - \frac{2\pi}{5} - 2\pi l) + \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(w + \frac{2\pi}{5} - 2\pi l)$$

    That is for each period (0 to 2π) we have:

    $$X[e^{jw}] = \pi\delta(w - \frac{2\pi}{5}) + \pi\delta(w + \frac{2\pi}{5}) \quad where \quad -\pi \le w < \pi$$

    So the whole signal is shown below:



5.3. Fourier Transform Convergence

In the Discrete-Time Fourier Transform case, convergence is required in order for Fourier Transform to exist. This is another similarity between the Continuous-Time and Discrete-Time Fourier Transform. Here again we have to deal with infinite summation to find the Fourier Transform. So we need to ensure that the signal, x[n], is convergent before using the following Fourier Transform equation to find the Fourier Transform of x[n].

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$ Fourier Transform Equation

As discussed earlier, in order for x{n] to be convergent and have a Fourier Transform, the following must be true:

$$\sum_{n=-\infty}^{\infty} |\, x[n]\,| < \infty$$ which says the x{n] is absolutely summable

or

$$\sum_{n=-\infty}^{\infty} |\, x[n]\,|^2 < \infty$$ which says that the sequence x{n] has finite energy

In this case unlike the Continuous-Time situation we do not expect to see any behavior similar to the Gibbs Phenomenon since limits are finite. You may recall that the Gibbs Phenomenon says that as the period increases, the ripples in the partial sums become compressed toward the discontinuity, but for any finite value of period, the peak amplitude of the ripples remains constant.

Further, not all LTI systems have impulse responses that converge. For example $2^n u[n]$ does not converge since $\sum_{n=-\infty}^{\infty} |\, h[n]\,|\ is\ not < \infty$ or diverges which means that it does not have a Fourier transform.

We will introduce an extension to Discrete-Time Fourier Transform called z-transform that will allow us to use the transformation techniques for LTI systems for which the frequency response does not converge.

Finally, the issue of convergence does not exist for Inverse Fourier transform since the limits are defined within any 2π period:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}\,dw$$ Inverse Fourier Transform

5.4. Properties of the Discrete-Time Fourier Transform

The development of Fourier transform $X(e^{jw})$ provides us with a frequency description of signal (frequency-domain) while x[n] provides us with time description of signal (time-domain). The following table presents the properties of Discrete-Time Fourier Transform:

Properties of Discrete-Time Fourier Transform of Aperiodic Signal

Properties	**Aperiodic Signal x[n] and y[n]**	**Fourier Transform: $X(e^{jw})$ and $Y(e^{jw})$ "always periodic with period 2π"**
Linearity	Ax[n] + By[n]	$AX(e^{jw}) + BY(e^{jw})$
Time Shifting	$x[n - n_0]$	$e^{-jwn_o} X(e^{jw})$
Frequency Shifting	$e^{jw_o n} x[n]$	$X(e^{j(w-w_o)})$
Conjugation	x*[n]	$X^*(e^{-jw})$
Time Reversal	x[-n]	$X(e^{-jw})$
Time Expansion	$x_{(k)}[n]$ = x[n/k] if n = multiple of k 0 if n ≠ multiple of k	$X(e^{jkw})$
Convolution	x[n]*y[n]	$X(e^{jw})Y(e^{jw})$
Multiplication	x[n]y[n]	$\frac{1}{2\pi} X(e^{jw}) * Y(e^{jw}) \quad or$ $\frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\theta}) Y(e^{j(w-\theta)}) d\theta$
Differencing in Time	x[n] – x[n – 1]	$(1 - e^{-jw})X(e^{jw})$
Summation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-jw}} X(e^{jw})$
Differentiation in Frequency	nx[n]	$j \frac{dX(e^{jw})}{dw}$
Conjugate symmetry	Real x[n]	$X(e^{jw})=X^*(e^{-jw})$
Symmetry	Real and even x[n] Even x[n] Real and odd x[n]	$X(e^{jw})$ is real and even $X(e^{jw})$ is real $X(e^{jw})$ is purely imaginary and odd
Parseval's Relation Total Energy of the Aperiodic signal	$Total\ Energy = \sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(e^{jw}) \rvert^2\ dw$	

These properties are important to analysis and our ability to move between time-domain and frequency-domain representations. Access to the two domains allows us to perform analysis in the domain which is most efficient.

All the above listed properties can be proved by appling the following Fourier Transform and Inverse Fourier Transform equations:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$ Fourier Transform Equation

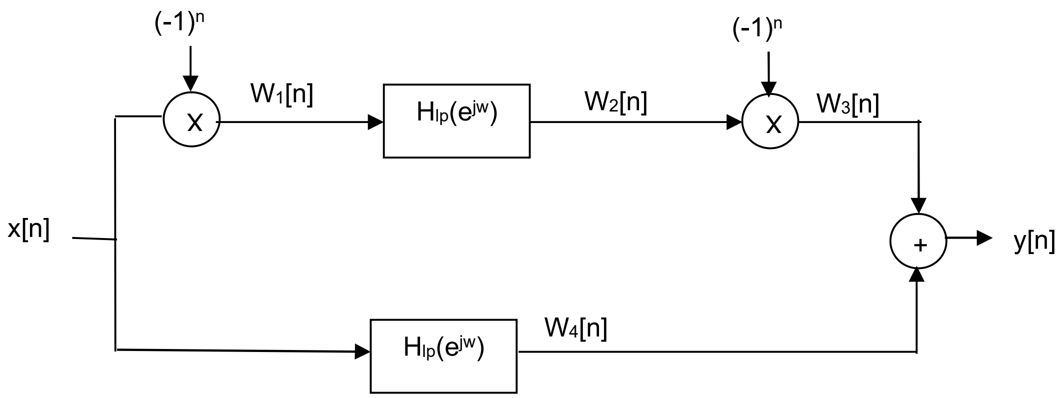
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$ Inverse Fourier Transform

Two important properties to consider:

- Periodicity: Discrete-Time Fourier Transform is always periodic in w with period 2π for all x[n]
  $X[e^{jw+2\pi}] = X[e^{jw}]$
- Convolution:
  where y[n]=x[n]*h[n] and $y(e^{jw}) = X(e^{jw})\,H(e^{jw})$

❖ Example - Discrete-Time Fourier Transform Properties

- Example 1: Consider the following LTI system where the frequency response $H_{lp}(e^{jw})$ are ideal low pass filters with cutoff frequencies $\pi/4$ and unity gain in the pass-band.



Find the frequency response of the whole system.

<u>Solution</u>

- First work through the top path.
  * We know that $(-1)^n = e^{j\pi n} = (\cos(\pi n) - j\sin(\pi n))$

  * We also know that this is a linear system therefore and applying frequency shifting property
  $e^{jw_0 n}x[n] \rightarrow X(e^{j(w-w_0)})$
  $W_1(e^{jw}) = X(e^{j(w-\pi)}$
  $W_2(e^{jw}) = W_1(e^{jw})H_{lp}(e^{jw}) = X(e^{j(w-\pi)})H_{lp}(e^{jw})$
  $W_3(e^{jw}) = W_2(e^{j(w-\pi)}) = X(e^{j(w-2\pi)})H_{lp}(e^{j(w-\pi)}) = X(e^{jw})H_{lp}(e^{j(w-\pi)})$

- Now the bottom branch
  $W4(e^{jw}) = X(e^{jw})H_{lp}(e^{jw})$

- Now the final addition of top and bottom branch:

  $$Y(e^{jw}) = W_3(e^{jw}) + W_4(e^{jw}) = X(e^{jw})H_{lp}(e^{jw}) + X(e^{jw})H_{lp}(e^{j(w-\pi)})$$

  $$Y(e^{jw}) = X(e^{jw})\left[H_{lp}(e^{jw}) + H_{lp}(e^{j(w-\pi)})\right]$$

  Therefore

  $$H(e^{jw}) = H_{lp}(e^{jw}) + H_{lp}(e^{j(w-\pi)})$$

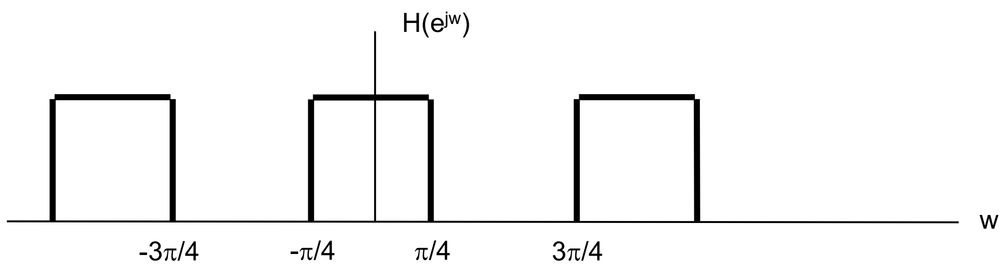
- Finally:

  * We know $H_{lp}(e^{jw})$ is a low pass filter with a cut-off frequency of π/4

  Therefore

  * $H_{lp}(e^{j(w-\pi)}) \to (-1)^n h_{lp}[n]$ which is the impulse response of the high pass filter with cut off frequency of 3π/4

  Based on the above two facts the system is a band pass filter with cut off frequencies at π/4 and -π/4 as shown below:



The following table summarizes some of the most common Discrete-Time Signals with corresponding Discrete-Time Fourier Transform and Fourier Series Coefficient.

Discrete-Time Fourier Transform and Fourier Series Coefficient

Signal, $x[n] = F^{-1}\{X(jw)\}$ Period N, $w_0 = 2\pi m/N$	**Fourier Transform $X(e^{jw}) = F\{x[n]\}$**	[1]**Fourier Series Coefficient, a_k**
$\sum_{k=<N>} a_k e^{jkw_0 n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(w - kw_0)$	$\frac{1}{N} \sum_{n=<N>} x[n] e^{jkw_0 n}$
$e^{jw_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(w - w_0 - 2\pi l)$	If $w_0 = 2\pi m/N$ $a_k = 1$ for k= .. m-N, m, m+N, 0 otherwise if $w_0/2\pi$ irrational → the signal is Aperiodic
$\cos(w_0 n)$	$\pi \sum_{l=-\infty}^{\infty} [\delta(w - w_0 - 2\pi l) + \delta(w + w_0 - 2\pi l)]$	If $w_0 = 2\pi m/N$ $a_k = 1$ for k= .. m-N, m, m+N, 0 otherwise If $w_0/2\pi$ irrational → the signal is Aperiodic
$\sin(w_0 n)$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} [\delta(w - w_0 - 2\pi l) - \delta(w + w_0 - 2\pi l)]$	If $w_0 = 2\pi m/N$ $a_k = 1/2j$ for k= ..m-N, m, m+N… -1/2j for k= ..-m-N, -m, -m+N… 0 otherwise If $w_0/2\pi$ irrational → the signal is Aperiodic
1	$2\pi \sum_{l=-\infty}^{\infty} \delta(w - 2\pi l)$	$a_0 = 1$ for k = 0, ±N, ±2N, … 0 otherwise
Periodic Square Wave x(t) = 1 |n| <N_1 0 N_1< |n| ≤ N/2 with Period N.	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(w - \frac{2\pi k}{N})$	If k = 0, ±N, ±2N, … $a_k = \frac{2N_1 + 1}{N}$ Otherwise $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[(\pi k/N)]}$
Pulse Train $\sum_{n=-\infty}^{+\infty} \delta(n - kN)$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(w - \frac{2\pi k}{N})$	$a_k = 1/N$ for all k
$a^n u[n]$ |a| < 1	$\frac{1}{1 - ae^{-jw}}$	N.A.
x[n] = 1 |N| ≤ N_1 0 |N| > N_1	$\frac{2\sin w(N_1 + 1/2)}{\sin(w/2)}$	N.A.

$\dfrac{\sin Wn}{\pi n}$ 0 < W < π	1 for 0 ≤ |w| ≤ W 0 for W < |w| > π $X(e^{jw})$ is periodic with period of 2π	N.A.
$\delta[n]$	1	N.A.
u[n]	$\dfrac{1}{1-e^{-jw}}+\sum_{k=-\infty}^{\infty}\pi\delta(w-2\pi k)$	N.A.
$\delta[n-n_0]$	e^{-jwn_0}	N.A.
$(n+1)a^n u[n]$ |a|<1	$\dfrac{1}{(1-ae^{-jw})^2}$	N.A.
$\dfrac{(n+r-1)!}{n!(n-1)!}a^n u[n]$ |a|<1	$\dfrac{1}{(1-ae^{-jw})^r}$	N.A.

Note: [1]*Fourier Series Coefficient exists only if signal is periodic.*

5.5. Summary of Fourier Series and Transform Equations

This section is a summary of key concepts from this chapter.

- Discrete-Time Fourier Transform
  x(t) may be aperiodic but must be converging.

  - Fourier Transform - Analysis Equation
    $X(e^{jw})$ is the Fourier Transform of x[n] and is periodic in frequency domain.

    $$X(e^{jw}) = F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

  - Fourier Inverse Transform - Synthesis Equation
    x(t) is the Fourier Inverse Transform of $X(e^{jw})$ and is aperiodic in frequency domain.

    $$x[n] = F^{-1}\{X(e^{jw})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- Summary of Phase Transform, Fourier series and Fourier Transform

Transform	Time Domain (t)	Frequency Domain (w)
Phasor Transform	x(t) a sinusoidal signal in steady state system.	
Fourier Series	x(t) is periodic CT Signal x[n] is periodic DT Signal	a_k is aperiodic a_k is periodic
Fourier Transform (s=jw)	x(t) is aperiodic CT Signal x[n] is aperiodic DT Signal	X(jw)=F{x(t)} is aperiodic $X(e^{jw})=F\{x[e^{jw}]\}$ is periodic
Laplace Transform (s=jw+σ)	x(t) is aperiodic CT Signal	X(jw)=L{x(t)}
Z Transform (s=jw+σ)	X[n] is aperiodic DT Signal	X(z)=Z{x[n]}

- The following table summarizes the Fourier Series and Transform relationships.

	Continuous time Time Domain	Continuous time Frequency Domain	Discrete time Time Domain	Discrete time Frequency Domain
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}$ *Periodic in time*	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jkw_0t}dt$ *Aperiodic in freq.*	$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$ *Periodic in time*	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}$ *Periodic in frequency*
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$ Aperiodic in time	$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$ Aperiodic in freq.	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$ Aperiodic in time	$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$ Periodic in frequency

5.6. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 5.

- Lathi, B. Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 3.

- Stremler, F*. Introduction to Communication Systems* (1990) Addison-Wesley Publishing Company
  Chapter 3.

- Nilsson, J. Electrical Circuits. (2004) Pearson.
  Chapter 16 and 17.

5.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

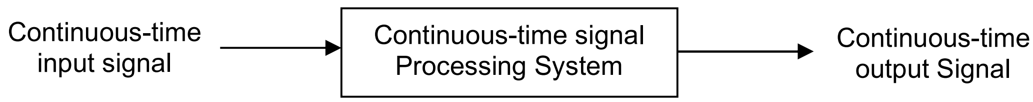
Chapter 6. Sampling

Key Concepts and Overview

- Introduction
- Sampling Theorem
- Aliasing Caused by Under Sampling
- Interpolation Techniques for Signal Reconstruction From Samples
- Statistical Approach to Sampling
- Additional Resources

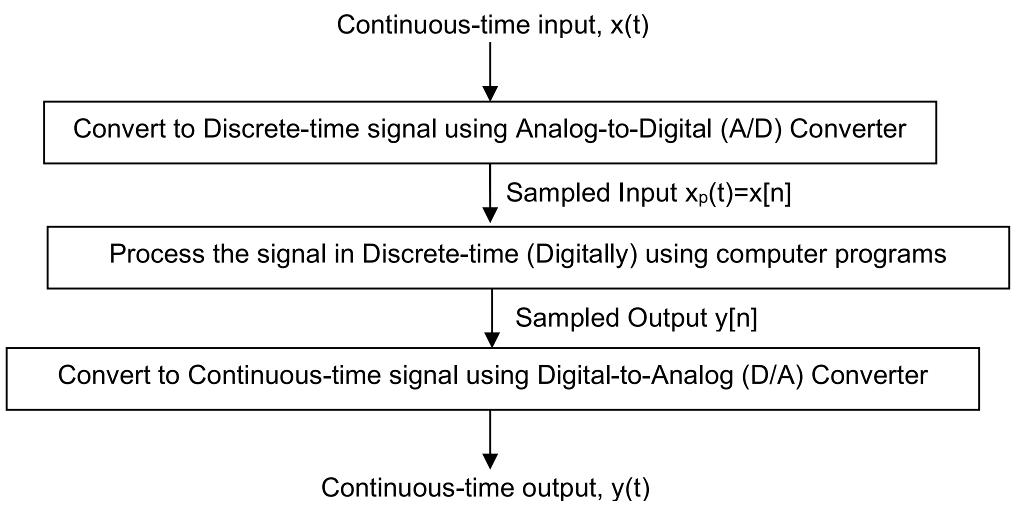
6.1. Introduction

This chapter focuses on sampling theory and its application to signals and systems. Natural signals such as images/light, sound/air pressure and movement are all Continuous-Time signals. This is of course true both in the sensing of this signal as well as control/generation of these types of signals. Prior to the advent of high speed computers and Fast Fourier Transform techniques, systems were designed to process the Continuous-Time signals directly, producing Continuous-Time output as depicted by the following diagram:



A filter build with active or passive electrical component is a good example of traditional Continuous-Time processing system. The flexibility of such systems are limited in-term of their ability to adapt to changing requirements of bandwidth, pass and no-pass transition time and cost. Additionally, it would be difficult to build systems that behave ideally with transitions that are instant such as in ideal filters.

On the other hand, using the ability to convert the Continuous-Time (analog) input signal to Discrete-Time (digital) signals then processing it in Discrete-Time (digitally) and converting it back to Continuous-Time output (analog), enables designer to reduce cost while increase flexibility and precision. The following diagram graphically describes such a system.



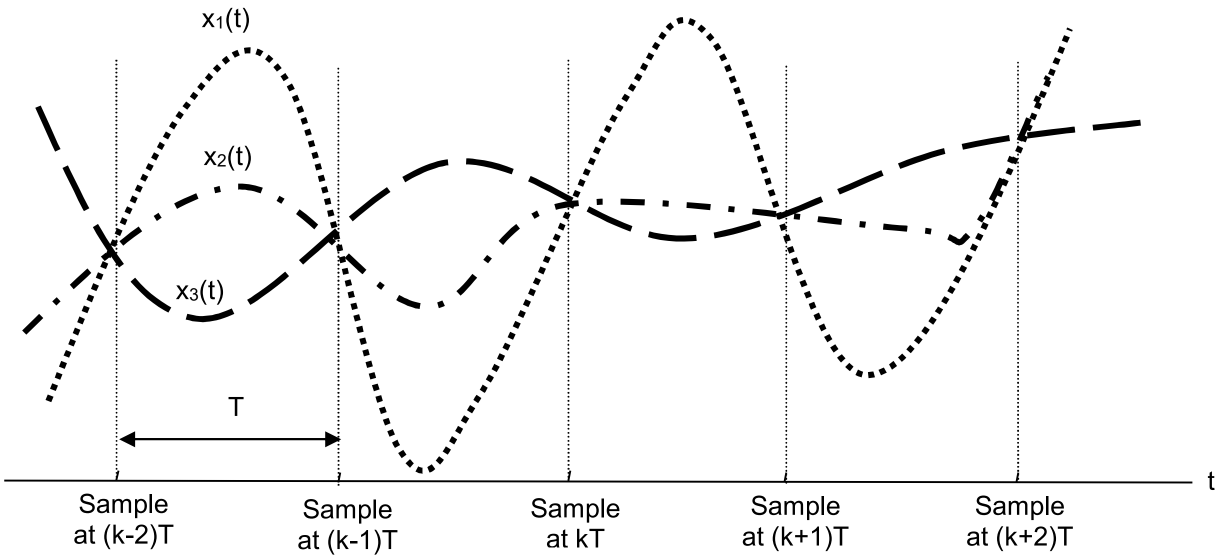
With the advent of high speed computers and Fast Fourier Transforms techniques, this process delivers a more cost effective, flexible and precise system than direct Continuous-Time (analog) approach. Today, systems are designed using this approach except in cases where some unusual circumstance may exist. An example of unusual circumstance is extreme high conversions (sampling rate) requirements. Sampling rates of Giga samples per second is considered too high for this approach and it would be best designed in direct continuous-time.

The key question that this chapter is concerned with is the conversion from continuous to Discrete-Time and back such that the original signal can be uniquely reconstructed. Sampling is the process of taking values of Continuous-Time signal at equal intervals and using them to construct a discrete time signal representing the original signal. First question to be answered in this chapter is how many samples per seconds are required for complete reconstruction which leads to the Sampling Theorem.

In the next section, Sampling Theorem is introduced. The Sampling Theorem is a critical element in system design since it is the bridge between Continuous-Time and discrete-time.

6.2. Sampling Theorem

Let's start by accepting the fact that a signal may not be uniquely specified by a sequence of equally spaced samples. For example, it is clear that $x_1(t) \neq x_2(t) \neq x_3(t)$ but the samples at equally spaced intervals are equal. $x_1(kT) = x_2(kT) = x_3(kT)$ as shown below:



However if the signal is band-limited which means the value of signal above a finite frequency is equal to 0, and we sample the signal at short enough intervals then the samples uniquely represent the signal. Let's explore the previous sentence in more detail:

First, signal is band-limited. This means that maximum frequency of the signal may be represented with a finite value W_M. In other words, the Fourier Transform of signal at frequencies above W_M is equal to 0.

Second, the sample must be high enough frequency (short intervals) which we show latter means that the sample frequency, W_S, must be more than twice the maximum signal frequency, W_M . This key concept is known as Sampling Theorem. If Sampling Theorem conditions are met then the samples uniquely specify the signal, and we can reconstruct it perfectly.

Now, let's focus on the process of sampling a Continuous-Time signal. You may recall from earlier chapters, the concept of an Impulse Train Signal, p(t):

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

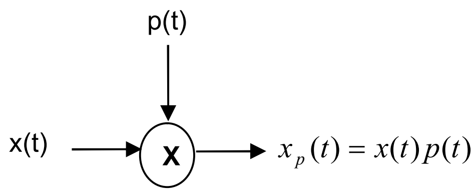
The impulse train signal is used to sample the signal. Basically, regularly spaced impulses are applied to the signal x(t) to obtain the sampled sequence as shown below:

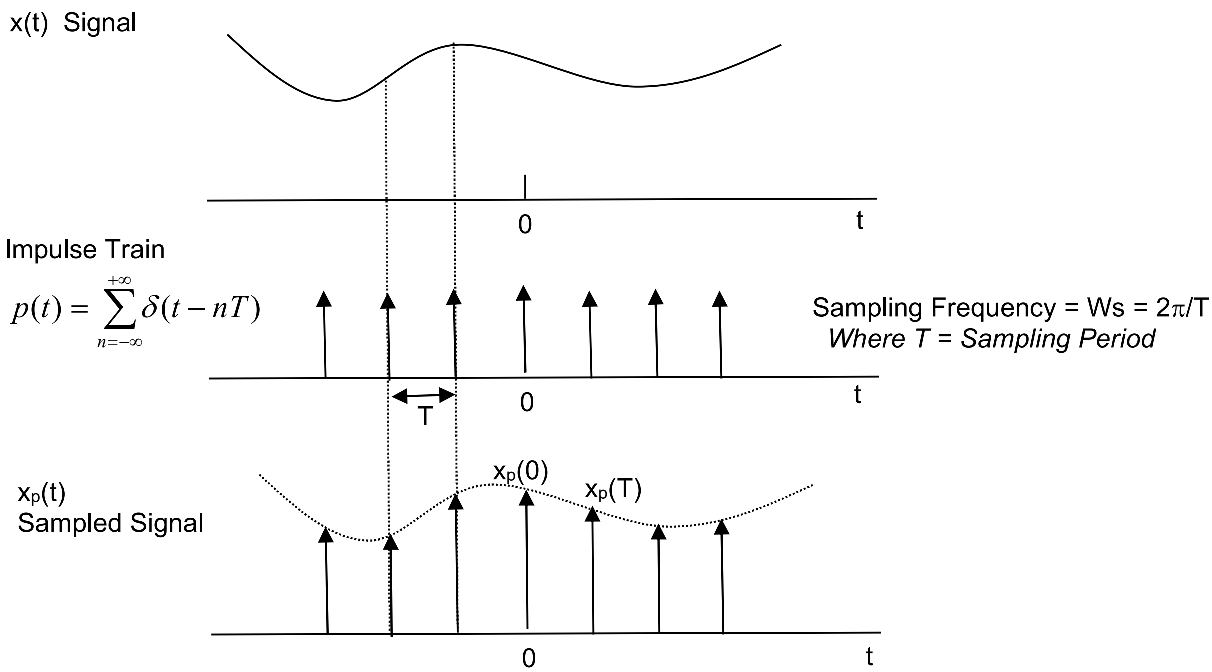
$$x_p(t) = x(t)p(t) \text{ where Impulse Train } p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Therefore:

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{+\infty} x(t)\delta(t-nT) = \sum_{n=-\infty}^{+\infty} x[nT]$$

Remember the sifting property of impulse function which implies $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$. The following diagram graphically represents the sampling process and the process used to generate the sampled signal, $x_p(t)$:





Therefore the equation describing the $x_p(t)$ may be written as:

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)$$

- Example – For the signal x(t) = sin(2000πt) and sampling period, Ts = 0.4 msec:
  a) Write the equation for the impulse train, p(t), and sampled signal, x(t).
  b) Draw the signals x(t) and p(t).

  Solution:

Student Exercise.

Applying the Multiplication property from Continuous-Time Fourier Transform Property:

$$X_p(jw) = \frac{1}{2\pi} X(jw) * P(jw) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(w-\theta)) d\theta$$

Remember that the following two facts:

1) Fourier Transform of p(t) is $P(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - kw_s)$ *Note: Ws = 2π/T*

2) Convolution of impulse function and X(jw) results in a shifted signal X(jw) as shown below:

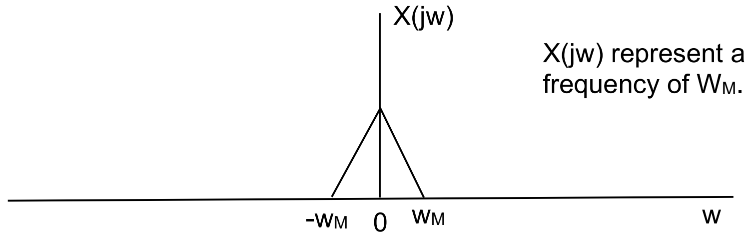
$$X(jw) * \delta(w - w_0) = X(j(w - w_0))$$

Using the above two facts in combination with X_p(jw), we can rewrite the sampled signal equation as:

$$\boxed{X_p(jw) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(w - kw_s))}$$

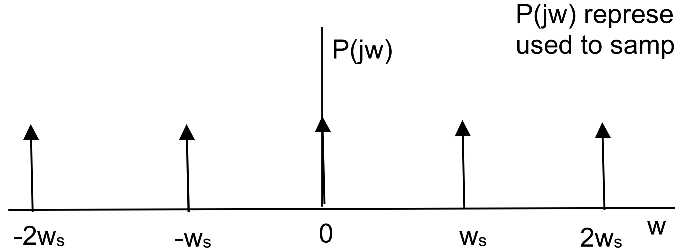
Now that we have the Fourier Transform of the sampled signal in terms of the Fourier Transform of the shifted original signal, we can explore the signal and sampling in frequency spectrum. This view provides the basis to discuss the implication of the equation:

$$X_p(jw) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(w - kw_s))$$



X(jw) represent a signal with Maximum frequency of W_M.

P(jw) represents a pulse train that will be used to sample at the frequency of Ws.

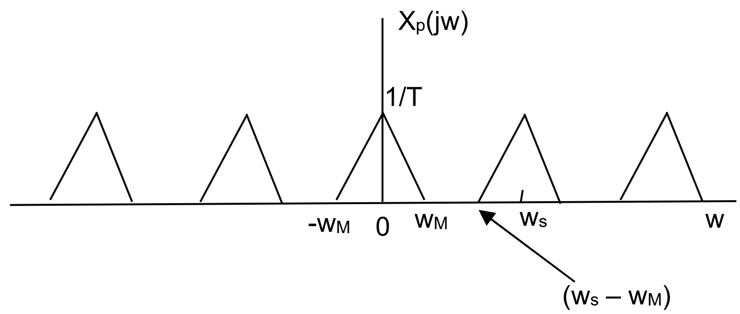


The signal Frequency W_M and Sampling Frequency W_s relationship determines if the original signal can be recovered from the sampled signal. There are two possible scenarios which are outlined below:

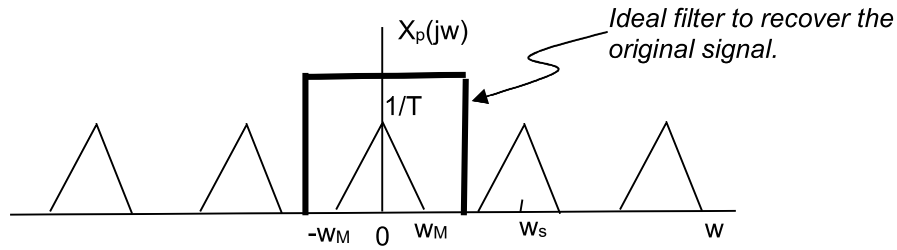
Scenario 1) The Maximum signal Frequency W_M and Sampling Frequency W_s relationship meeting the following condition:

$$W_M < (W_s - W_M) \rightarrow W_s > 2W_M$$

As shown in the following diagram, the sampled signal does not overlap with the harmonics which are the side effect of the sampling process. Therefore the original signal can be uniquely recovered from the sampled signal.



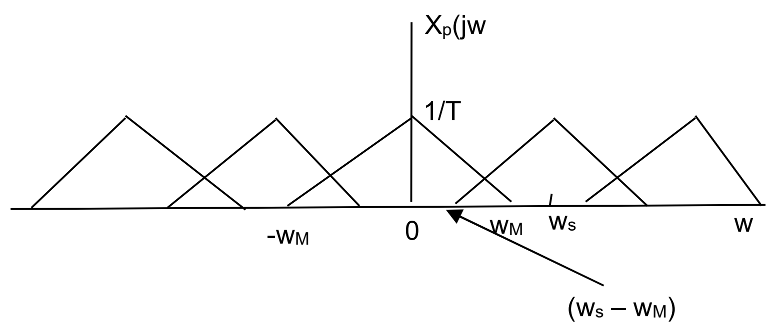
In this scenario, the signal can be recovered by using a low pass filter with a gain of T and cut-off frequency greater than W_M and less than (W_s - W_M).



Scenario 2) The Maximum signal Frequency W_M and Sampling Frequency W_s relationship meeting the following condition:

$$W_M \geq (W_s - W_M) \rightarrow W_s \leq 2W_M$$

As shown in the following diagram, the sampled signal overlaps with the harmonics of the original signal which results in a side effect refers to aliasing. This means that content of an alias (Harmonic) is corrupting the original signal spectrum. Therefore the original signal cannot be uniquely recovered from the sampled signal. Below is frequency spectrum of sampled signal under aliasing scenario:



This leads us to the Nyquist-Shannon Sampling Theorem or simply Sampling Theorem which are stated below:

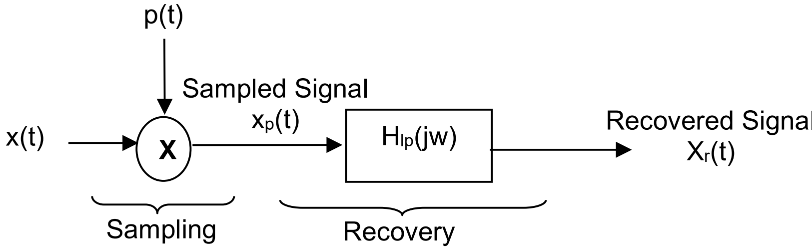
If the signal x(t) is band-limited ($X(jw)=0$ for $\lvert w\rvert > W_M$), then the signal x(t) is completely (uniquely) determined by its samples x(nT) only if $W_s > 2W_M$.

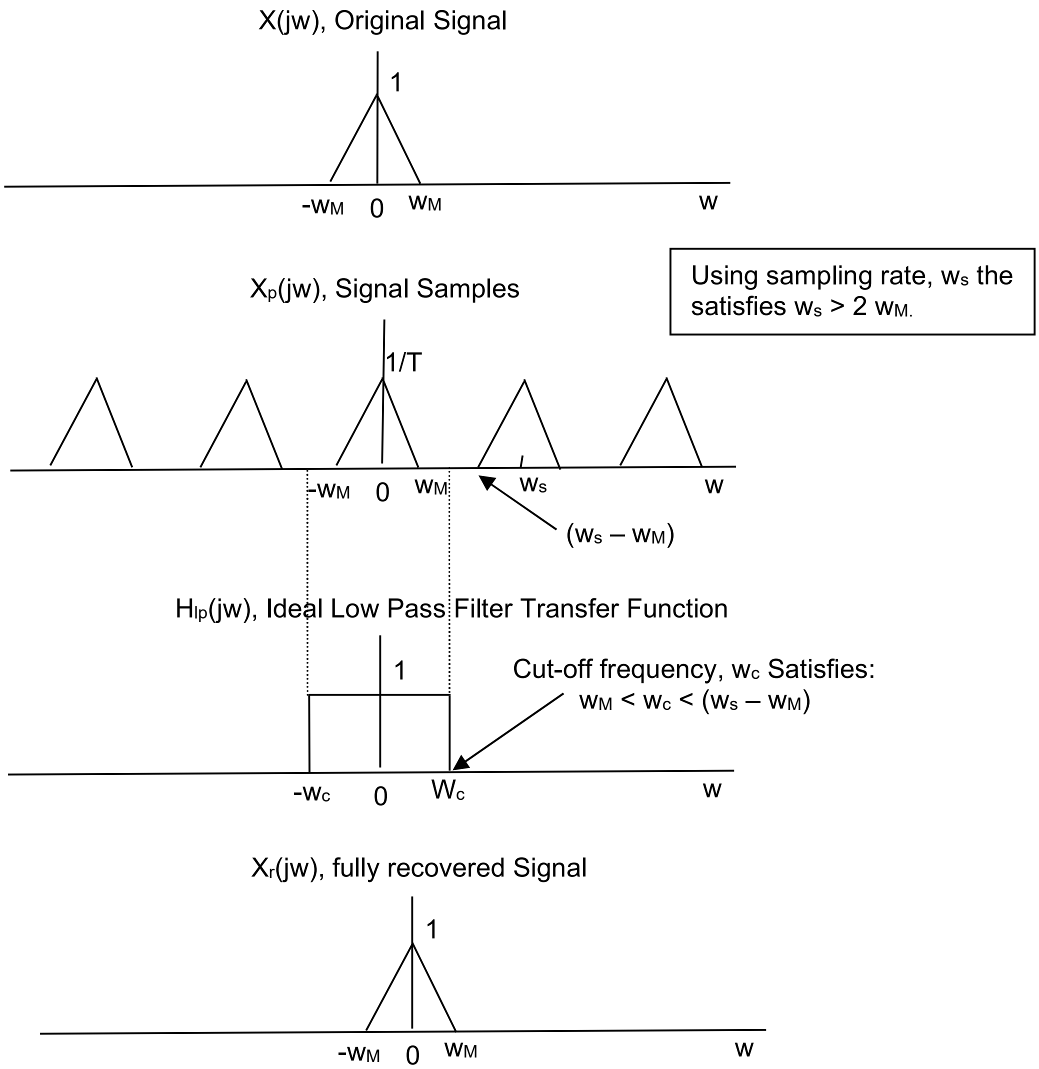
Notes:
[1] sampling frequency, Ws = $2\pi/T$.
[2] n=0, ±1, ±2, …

Nyquist Rate is equal to $2W_M$. Sampling Theorem requires that sampling frequency exceed Nyquist Rate ($2W_M$) for the sampled signal to be recoverable. ***Nyquist Frequency*** refers to W_M which is one half of the *Nyquist Rate*.

The process to sample a Continuous-Time signal under the Sampling Theorem can be accomplished by multiplying the Continuous-Time signal x(t) with an impulse train signal p(t) to generate the sampled signal $x_p(t)$. Assuming the condition of Sampling Theorem is met ($W_s > 2W_M$*) then the original signal can be uniquely reconstructed using an ideal low pass filter with cut-off frequency w_c between* W_M *& (*W_s- W_M*)*

Below is an example of the process used for sampling a continuous signal and then reconstructing the signal from the samples using a low pass filter (this example meets the sampling theorem requirement of ($W_s > 2W_M$):





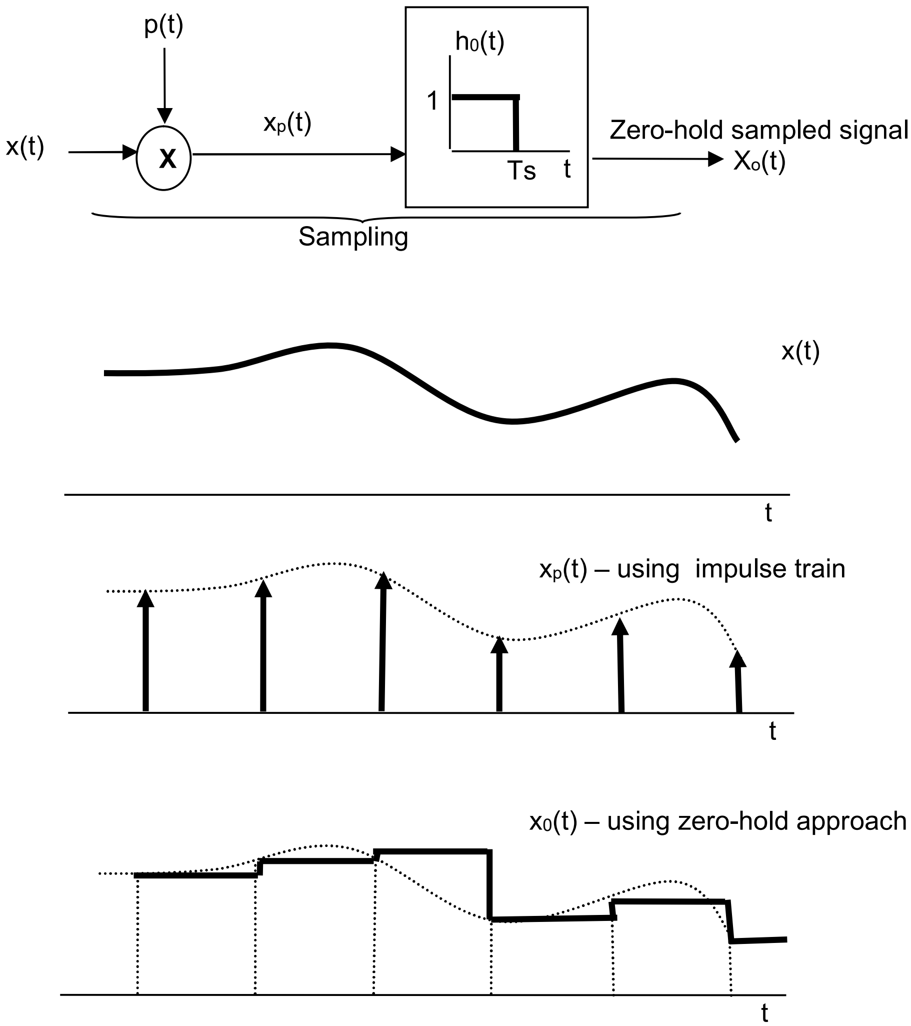
This example uses an ideal low pass filter to recover the signal. Since real filters will not have the exact characteristic of an ideal low pass filter which means that there may be distortion in the reconstructed signal depending on how closely the recovery low pass filter matches the characteristic of an ideal low pass filter. Just a quick reminder that ideal low pass filter meets the following conditions:

$H(jw) = 1 \quad \text{for } |w| < w_M$

$0 \quad \text{for } |w| > (w_s - w_M)$

The process of sampling described earlier requires the use of impulse train. Impulse function is a pulse of zero duration but finite amplitude. Of course, creating a signal that approaches impulse function definition is difficult even though the impulse function was useful in the process of arriving at the Sampling Theorem.

A more practical approach to sampling is a method called " Sampling with a zero-order hold". Zero-order hold system samples $x_0(t)$ at a given instant and holds that value until next instant at which another sample is taken. Here is an illustration:



- Example – For the signal, x(t), with only nonzero coefficients (a_0 -=-1, a_1=2, a_3=2j) with fundamental frequency of 22 Khz.

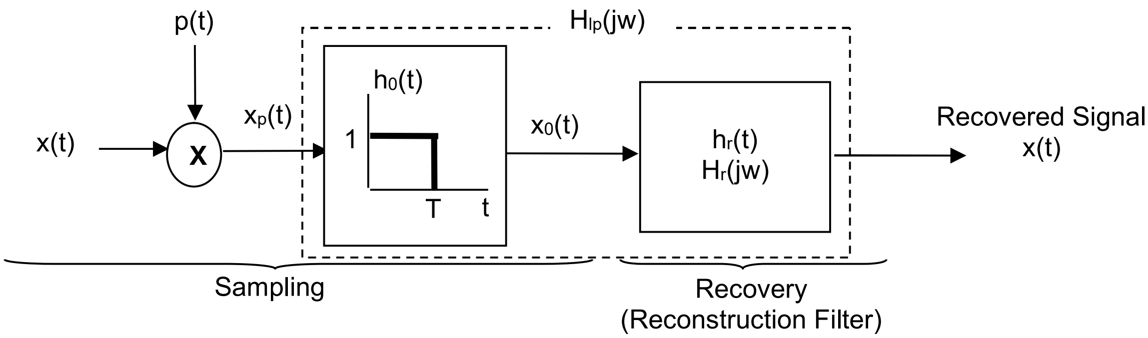
  a) What is the maximum frequency of signal x(t)?
  b) What are the nyquist frequency and rate?
  c) What is the Minimum and maximum sampling period so that x(t) is recoverable?
  d) What is the Minimum and maximum sampling frequency so that x(t) is recoverable?

  Solution:

- Example – What is the minimum sampling frequency, f_s, so that x(t)=sin(2000πt) is recoverable.

  Solution:

The Sampling and Recovery for the Zero-order hold system is as follows:



$h_0(t)$ is a square wave with duration from –T/2 to T/2 (shifted by T/2) therefore its Fourier Transform Function (using Fourier Transform Tables) can be written as:

$$H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

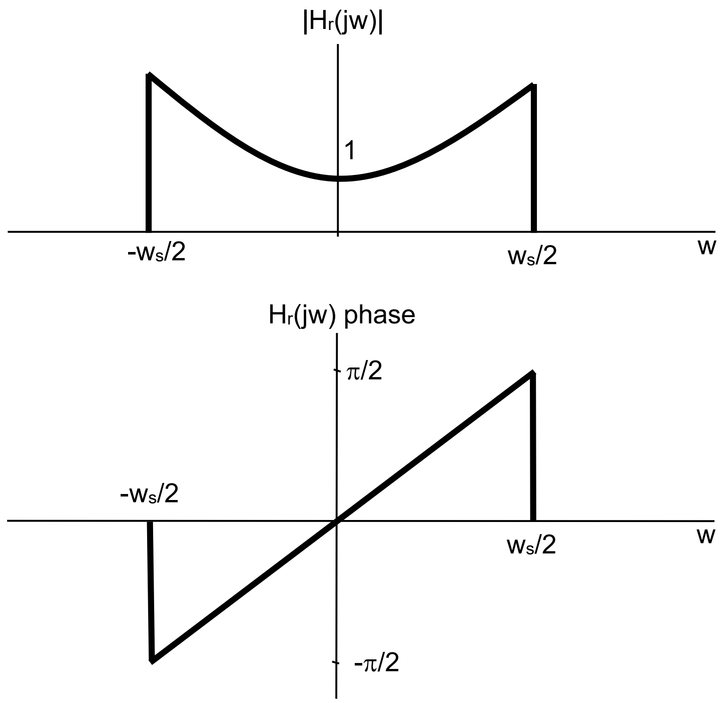
The next step is to find the frequency response for the reconstruction filter, $H_r(jw)$. As we saw earlier a low pass filter was sufficient to recover x(t) from $x_p(t)$ which means the following relationships has to be true:

$$H_{lp}(jw) = H_0(jw)H_r(jw) \rightarrow H_r(jw) = \frac{H_{lp}(jw)}{H_0(jw)}$$

Substituting the value of $H_0(jw)$ in the Hr(jw) results in the following equation:

$$H_r(jw) = \frac{e^{jwT/2}H_{lp}(jw)}{\frac{2\sin(wT/2)}{w}}$$

With the $H_{lp}(jw)$ cut off frequency equal to $W_s/2$, the ideal magnitude and phase for the reconstruction filter following a zero-order hold is shown below:



Remember, your actual result varies depending on how close your filter design and implementation can get to the ideal filter characteristics.

6.3. Aliasing Caused by Under Sampling

As Sampling Theorem states Sampling frequency (w_s) must at least two times the maximum signal frequency (w_M) in order to be able to completely reconstruct the original signal. In other words, the reconstructed signal $x_r(t)$ will be equal to the original signal. $w_s > 2w_M$

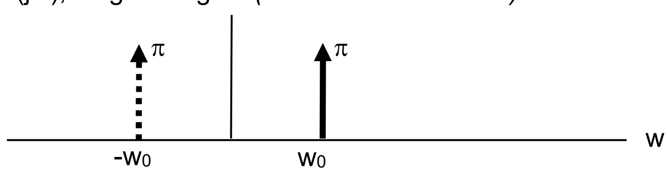
If above conditional is violated, $w_s \leq 2w_M$, then $x_r(t)$ is no longer equal to x(t), even though, the sample values are equal to the original signal at the point of sample { $x_r(nT)= x(nT)$}.

Let's use $x(t)=\cos(w_0t)$ to demonstrated the idea of aliasing. The Fourier Transform of x(t) can be written as:

$$X(jw) = \pi[\delta(w - w_0) + \delta(w + w_0)]$$

Based on the equation for the Fourier Transform, we can draw the frequency spectrum of $\cos(w_0t)$ as:

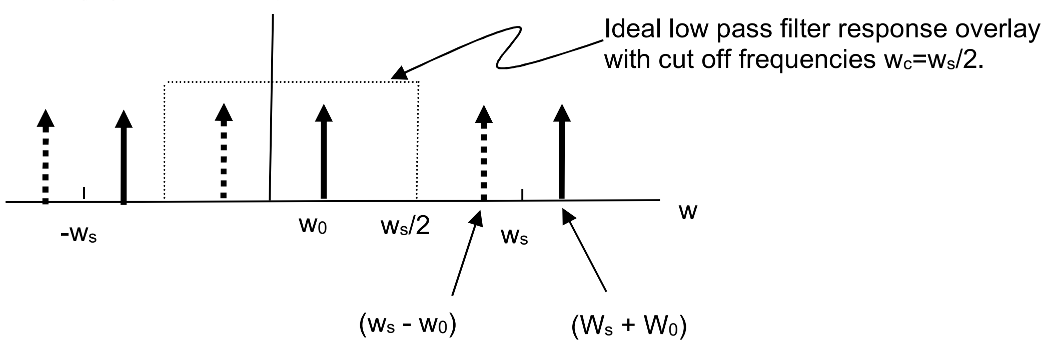
X(jw), Original Signal *(Dashed line is for –w)*



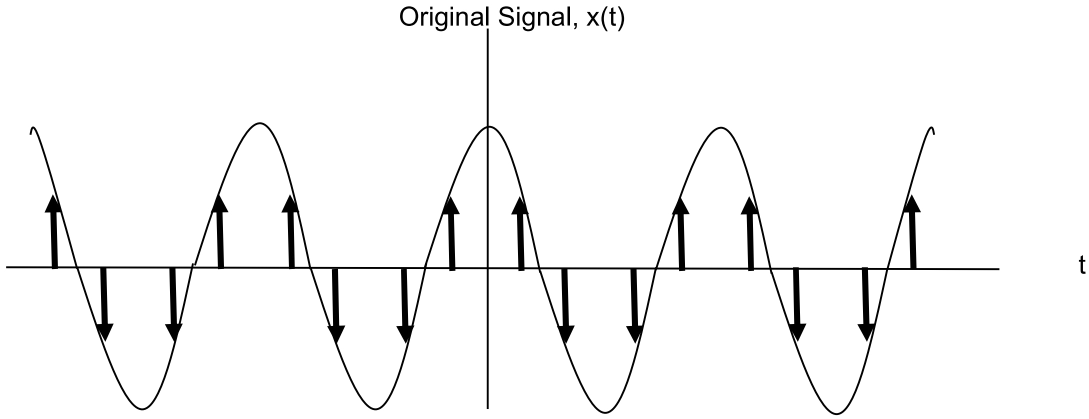
Depending on the Sampling frequency, we can have full recovery or aliasing defects in the reconstructed signal preventing full recovery. Below are examples of the two scenarios:

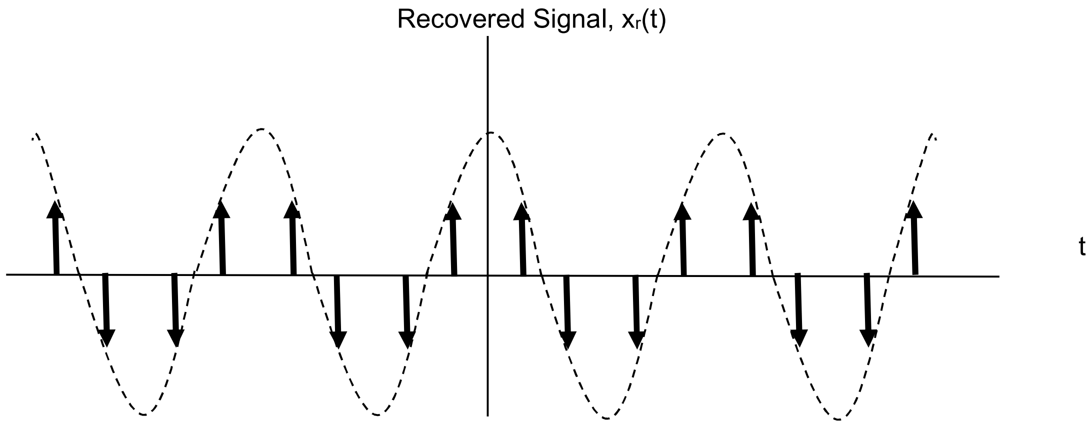
- No Aliasing Scenario - Sampled at frequency $w_s = 4\ w_o$
  In this case the maximum signal frequency is w_m which is equal to w_o therefore $w_s > 2\ w_m$ meeting the requirement of sampling theorem.

$X_p(jw)$, Sampled Signal $w_s=4w_0 =4\ w_M$



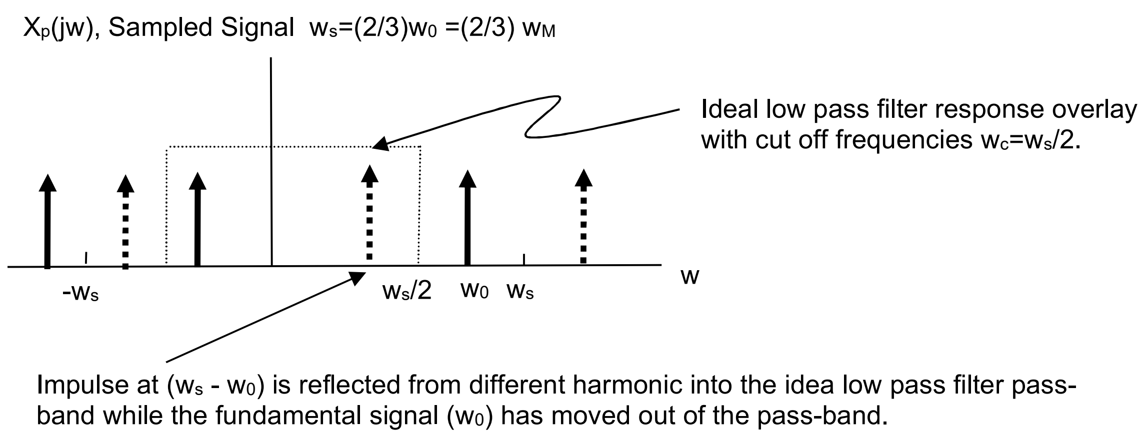
The time domain implication is that the original signal can be fully recoverable. Time-domain of original signal, sample impulses and recovered signal when w_s = (4) w_0. is shown below:



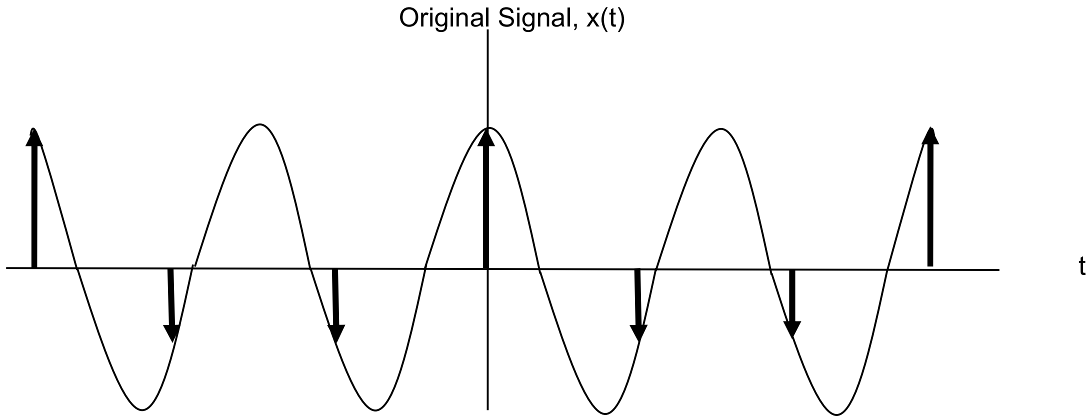


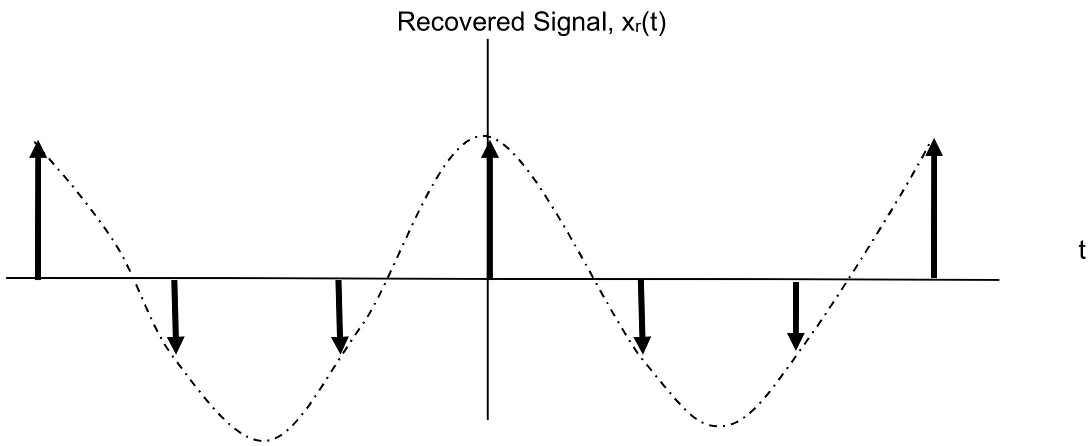
- Exercise - Students are encouraged to draw the time-domain of original signal, sample impulses and recovered signal when w_s = 8 w_0..

➢ Aliasing Scenario - Sampled at frequency w_s = (3/2) w_o
In this case the maximum signal frequency is also w_m which is equal to w_o therefore w_s < 2 w_m. Therefore the sampling frequency does not meet the requirement of sampling theorem.



In this case the original signal cannot be uniquely recovered. Time-domain of original signal, sample impulses and recovered signal when $w_s = (3/2)\, w_0$. is shown below:





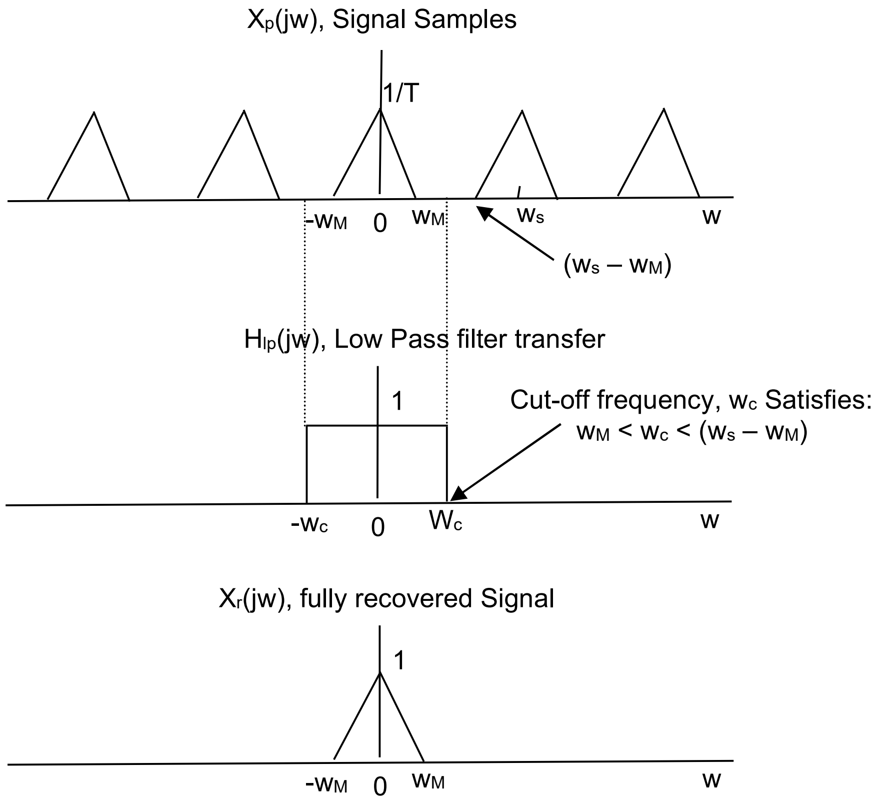
- Exercise - Students are encouraged to draw the time-domain of original signal, sample impulses and recovered signal when $w_s = 2\, w_0$.

➢ observations

As the w_0 increases the dashed lines move to the right and solid lines move to the left. If frequency is increased so that $w_0 > w_s/2$ then aliasing will occur. It is important to note that the sampling frequency must be greater than twice the maximum signal frequency; even equal results in aliasing.

6.4. Interpolation Techniques for Signal Reconstruction From Samples

In previous sections, we developed and discussed the Sampling Theorem and used it to sample Continuous-Time signals such that the sampled signal had sufficient information to fully recover the original signal from it. Further, it was shown that low pass filter would allow full recovery of the original signal without any aliasing as shown below again:



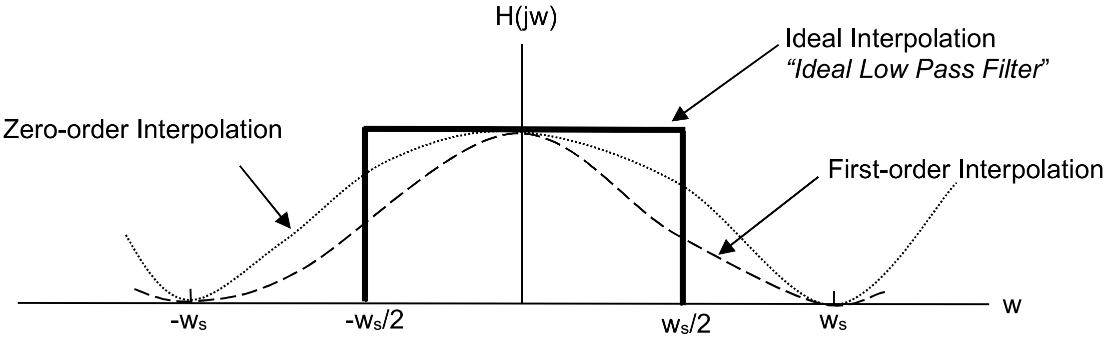
In other words, for band-limited signals, if the sampling rate is higher the twice the maximum frequency of signal, the signal can be constructed uniquely using Low Pass Ideal Filter.

The challenge is that although it is possible to approximate an ideal low pass filter, it is impossible to implement an ideal low pass filter. Interpolation is an attempt to fit a Continuous-Time signal to a set of samples that approximates the original function. Two of the simplest interpolations are:

1) zero-order hold which simply holds the last sampled value of the signal until it is time for the next signal sample.
2) one-order hold or linear interpolation uses a straight line from one sample to next sample to approximate the original signal.

   There are higher-order hold approaches which use higher order polynomials to approximate between the two samples.

One way to compare the two basic reconstruction approaches (zero-order hold and first-order hold) is by comparing their transfer function with an ideal low pass filter:



As it is shown in the above responses, both zero-order and first-order hold have potential for aliasing which may distorts the reconstruction.

By revisiting the process of ideal interpolating (low pass filter), we can describe the reconstructed signal $x_r(t)$ in terms of the ideal low pass filter characteristics as shown below:

$$x_r(t) = x_p(t) * h(t)$$

$$\textit{earlier we had } x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \textit{ therefore}$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)$$

$$\textit{This equation shows how to fit continous curve between samples points}$$

$$\textit{we know} \quad H_{LP}(jw) = 1 \textit{ for } |w| < w_c \textit{ otherwise } 0$$

$$\textit{therefore} \quad h(t) = \frac{w_c T \sin(w_c t)}{\pi w_c t}$$

Finally,

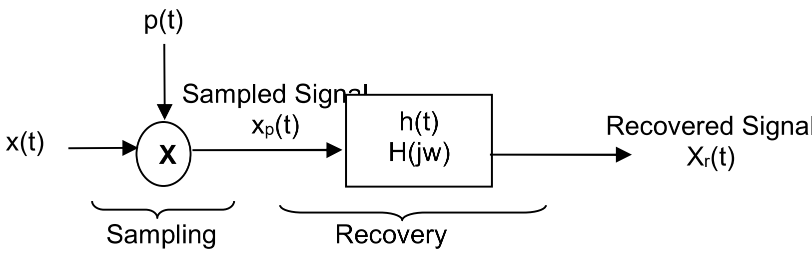
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T \sin(w_c(t - nT))}{\pi w_c (t - nT)}$$

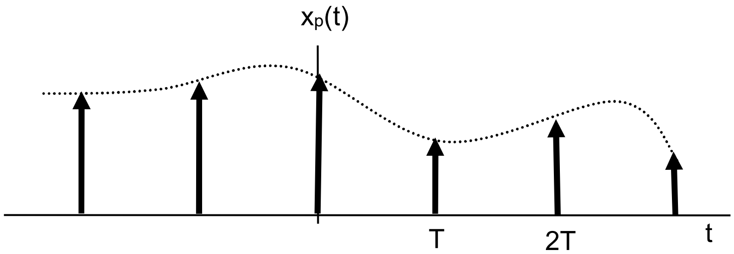
Interpolation using impulse response of low pass filter is commonly referred to as band-limited interpolation, since it implements exact reconstruction of x(t) is band limited and the sampling frequency satisfies the sampling frequency (ws>2wc).

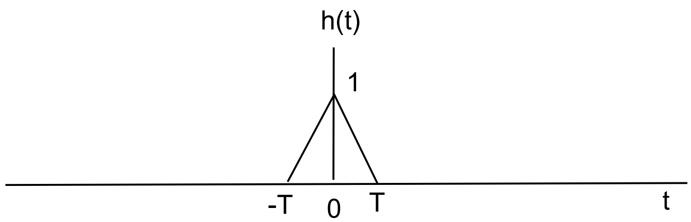
The zero-hold process was described in pervious section so at this point, we will explore the first-order hold which although more complex than zero-hold, it is simple enough to implement and provide improved approximation of the original signal:

First-order hold or Linear Interpolation which means that $X_p(t)$ is convoluted with h(t) that is linear in the form of a triangle. With the transfer function:

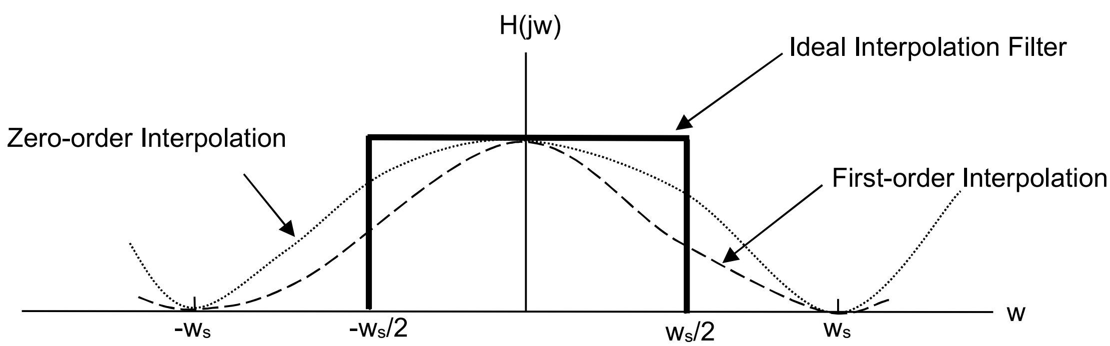
$$H(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$







A comparison of transfer functions for Ideal (ideal low pass), zero-order hold and first-order hold interpolation is shown below again:



6.5. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 7.

- Lathi B. Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 6.

6.6. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 7. Communication Systems

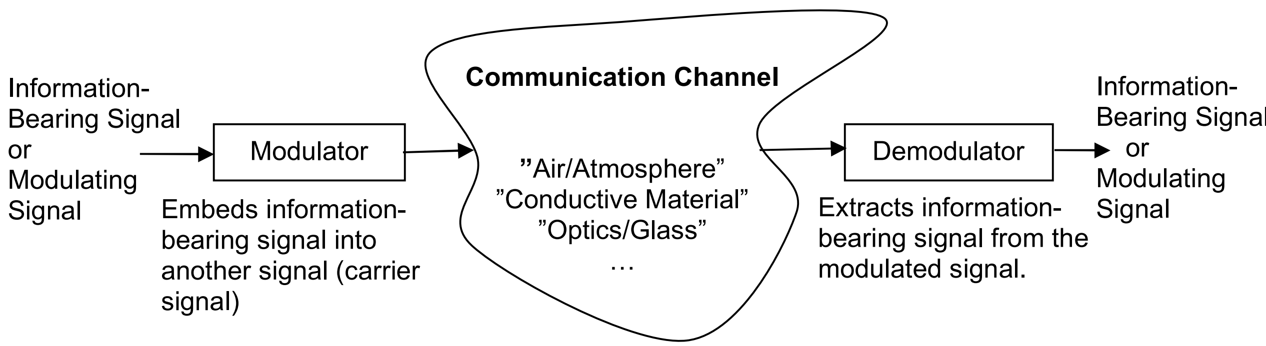
Key Concepts and Overview

- Introduction
- Amplitude Modulation (AM)
- Sinusoidal Amplitude Demodulation - Synchronous and Asynchronous
- Frequency-Division and Time Division Multiplexing
- Frequency Modulation(FM)
- Additional Resources

7.1. Introduction

Communication systems are integral part of every activity in the society. It would be challenging to think of any program that does not require the use of some type of communication system. Study of communication systems is a major field of electrical engineering and integral part of organizations in all section including Healthcare, transportation, technology and government.

Communication systems are designed to transmit information from one location to another. Most common means of transmission is Electro-Magnetic signals (electrical, light) through a communication channel as shown below:



Most communication channels are band-pass filters which are tuned to carry information in a specific range of frequencies. Modulator is part of transmitter and is used to convert the information-bearing signal such that it is tuned to the channel characteristic. The demodulator is part of receiver and is used to recover the information-bearing signal from the modulated signal. Another common reason for modulation/demodulation is to share the same physical channel to carry multiple signals across the same channel by dividing the frequency spectrum amongst the signals.

Two of the common modulation techniques used in communication system is:

- Amplitude Modulation (AM)
  Information-bearing signal is used to modulate the amplitude of another signal (carrier signal).

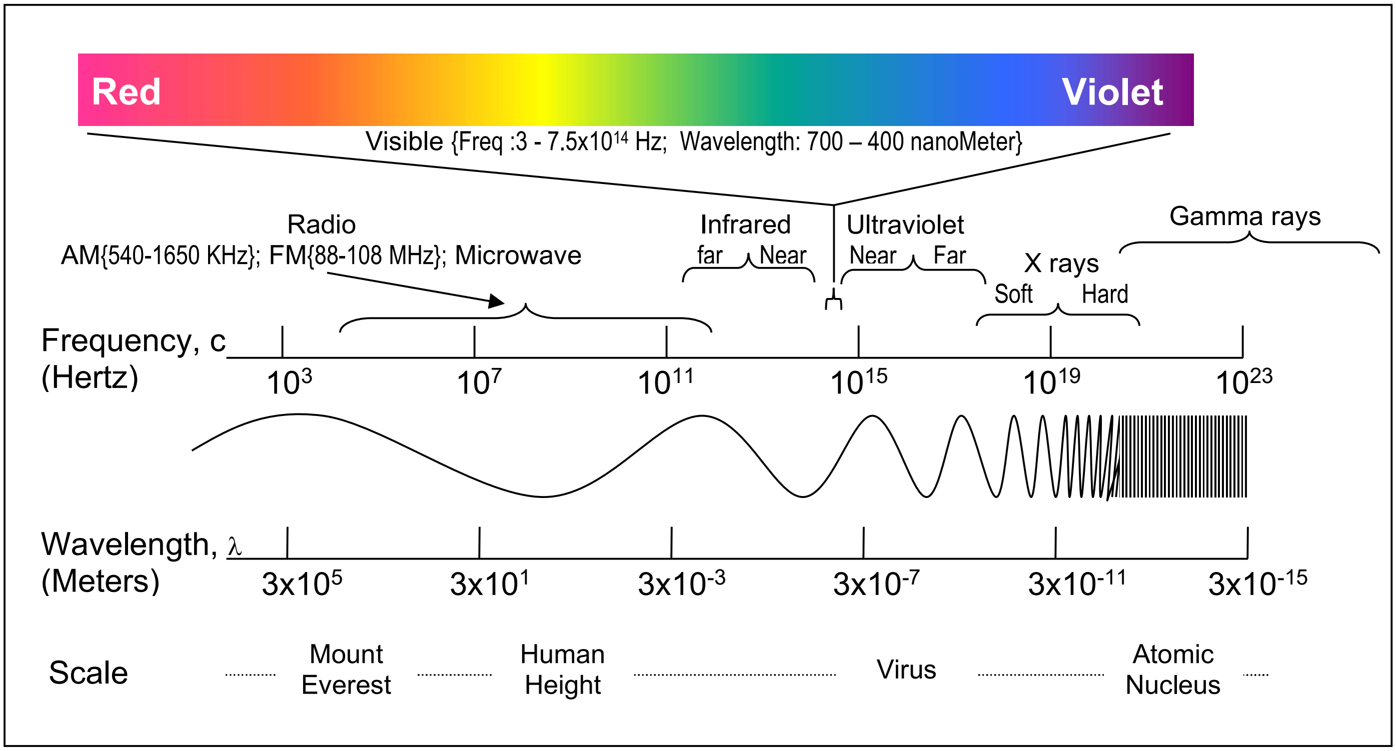
- Frequency Modulation (FM)
  Information-bearing signal is used to modulate the frequency of another signal (carrier signal)

One of the most common communication channels is the air/atmosphere. Frequency and wavelength are used to describe the ability of atmosphere to transmit signals across the electro-magnetic spectrum. The relationship between frequency and wavelength for an electro-magnetic signal is described by:

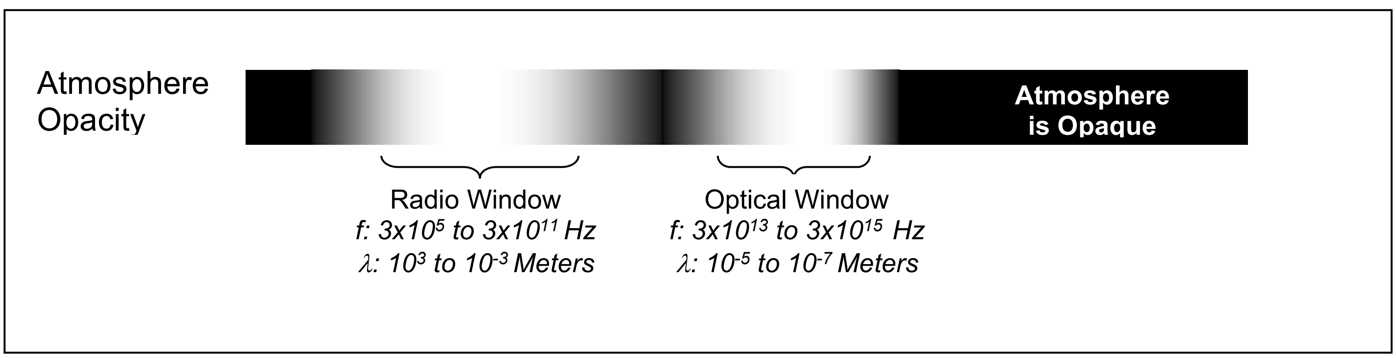
$\lambda = c/f$ where

- $c = 300\times10^6$ Meters/Sec (Speed of light)
- f = signal frequency in Hertz (Cycles/Sec)
- λ = signal wavelength in meters

The following chart shows the Electromagnetic spectrum and identifies few key ranges with their uses:

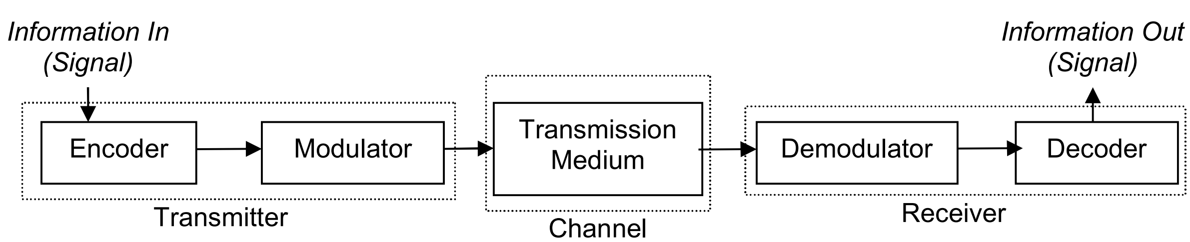


Below is the profile of atmosphere response or opacity to signals based on frequency range:



Other channels such as glass-fiber optics, metal-transmission lines or space have their own unique opacity profile (response) with respect to various signal frequencies.

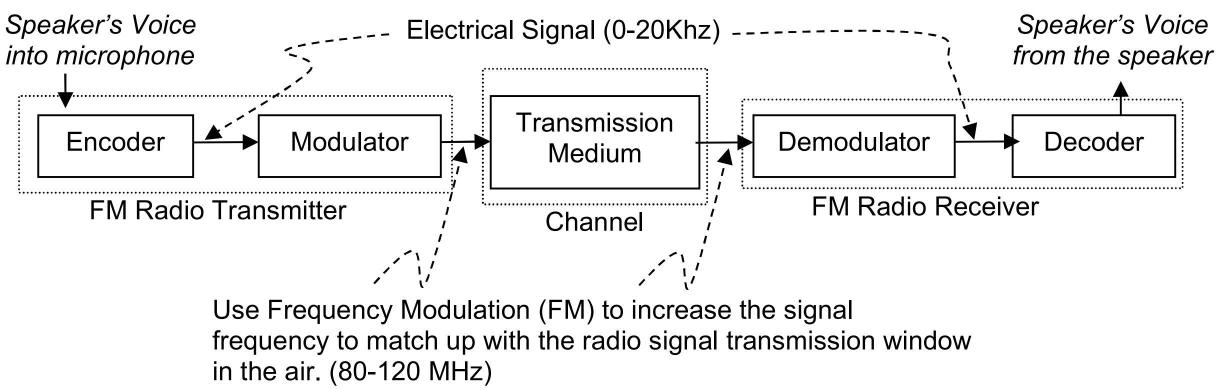
Let's start by considering a basic simplex system which is shown below with its components:



The word simplex refers to the fact that data only travels in one direction. The role of each component is described below:

Transmitter - Prepares the signal for efficient transfer over the channel
Encoder – Converts information into signal that is optimized for detection at the output
Modulator – Generates the modulated waveform to carry the Signal
Channel - Transmission medium (Air, glass-fiber optics, metal-transmission line, Space)
Receiver – Optimally extract the information from the channel
Demodulator – Extracts the signal from the modulated waveform
Decoder – Extracts the information from the signal received

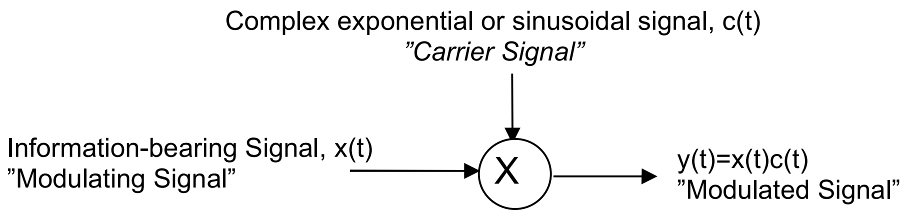
Frequency Modulated (FM) Radio is an example of a simplex system which is shown below:



The rest of this chapter explores these modulation techniques and their corresponding demodulation techniques.

7.2. Amplitude Modulation (AM)

As discussed in the introduction, modulation is used to move the signal to a frequency range that is available and effective in the selected communication channel. For example, AM radio is typically broadcasted in frequencies from 500 Khz to 2 Mhz. Human audible signal frequency range is within 20 Hz - 20 Khz range. So the recorded signal x(t) has to modulated using a carrier signal c(t) in order to generated modulated signal y(t)=x(t)c(t) for transmission. Below is system diagram of a Modulator:



In Amplitude Modulation (AM), the signal we want to transmit (Information-bearing signal) x(t) is used to modulate the amplitude of the carrier signal, c(t). The carrier signal typically is either a complex exponential signal or a sinusoidal signal as shown below:

$c(t) = e^{jw_c t}$	Complex exponential carrier signal
$c(t) = \cos(w_c t)$	Sinusoidal carrier signal

In both cases, w_c is the Carrier Frequency and the carrier phase is assumed to be zero for simplicity.

For the first case, let's use the complex exponential carriers and apply it to the information bearing signal in order to obtain the equation for the modulated signal:

$$Let\ c(t) = e^{jw_c t}\ \therefore$$

$$y(t) = x(t)e^{jw_c t}$$

$$Applying\ Fourier\ Transform\ multipication\ property$$

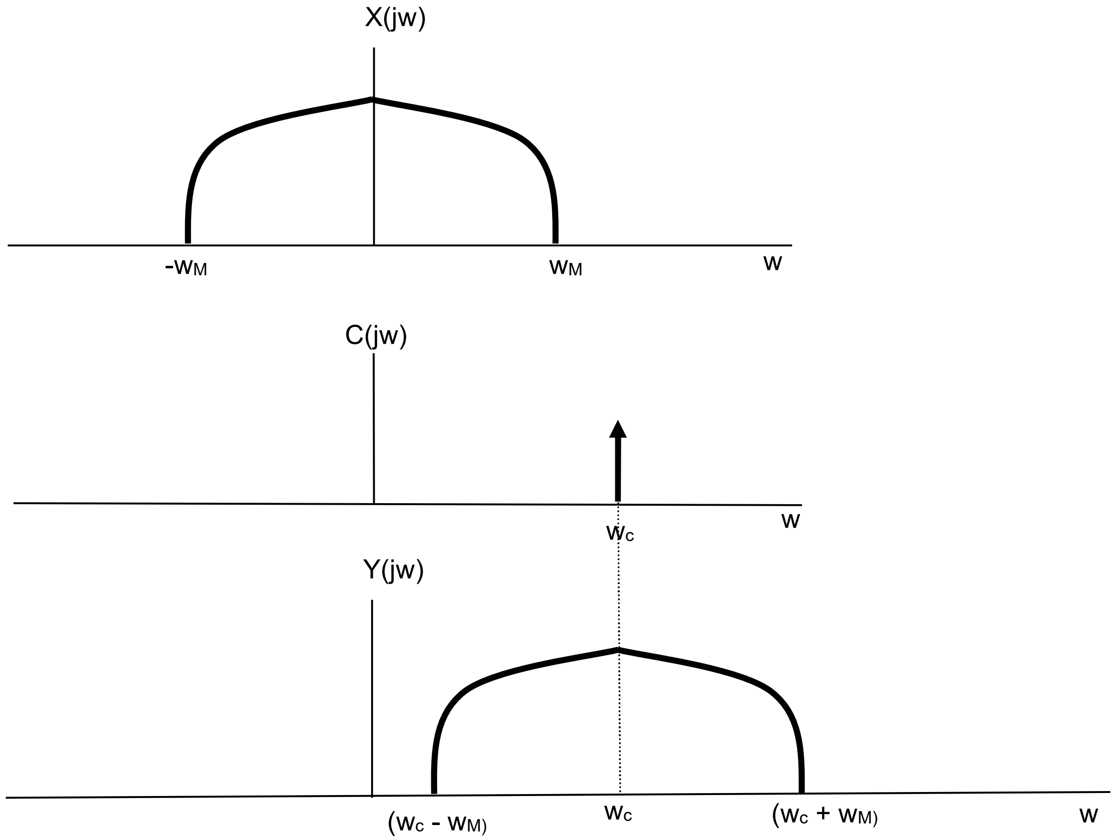
$$Y(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)C(j(w-\theta)d\theta$$

$$Given \rightarrow C(jw) = F\{c(t)\} = 2\pi\delta(w - w_c)$$

$$Applying\ the\ impluse\ to\ the\ above\ Convolution\ \mathrm{int}\,egration\ we\ get:$$

$$Y(jw) = X(j(w - w_c))$$

From the results, we can conclude that the modulated signal y(t) is information-bearing signal x(t) shifted by the carrier frequency w_c which is shown below graphically:



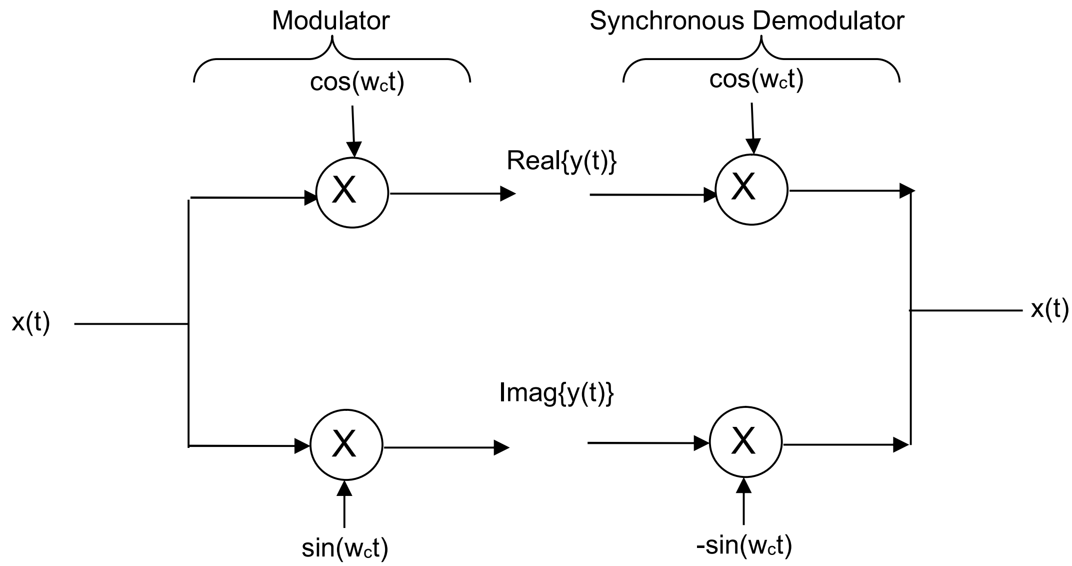
On the receiving end, the information bearing signal can be recovered by demodulating the signal which means shifting the signal back. Here are the mathematical equivalents of demodulation:

$$x(t) = y(t)e^{-jw_c t}$$

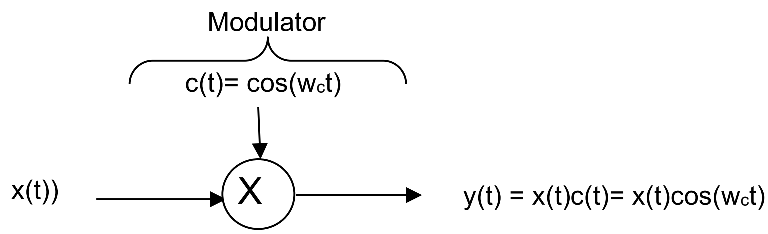
$$or$$

$$X(jw) = y(j(w + w_c))$$

The exponential carrier signal has real cosine and imaginary sine part (Application of Euler's relationship). Typically the actual implementation requires the use of the following system:



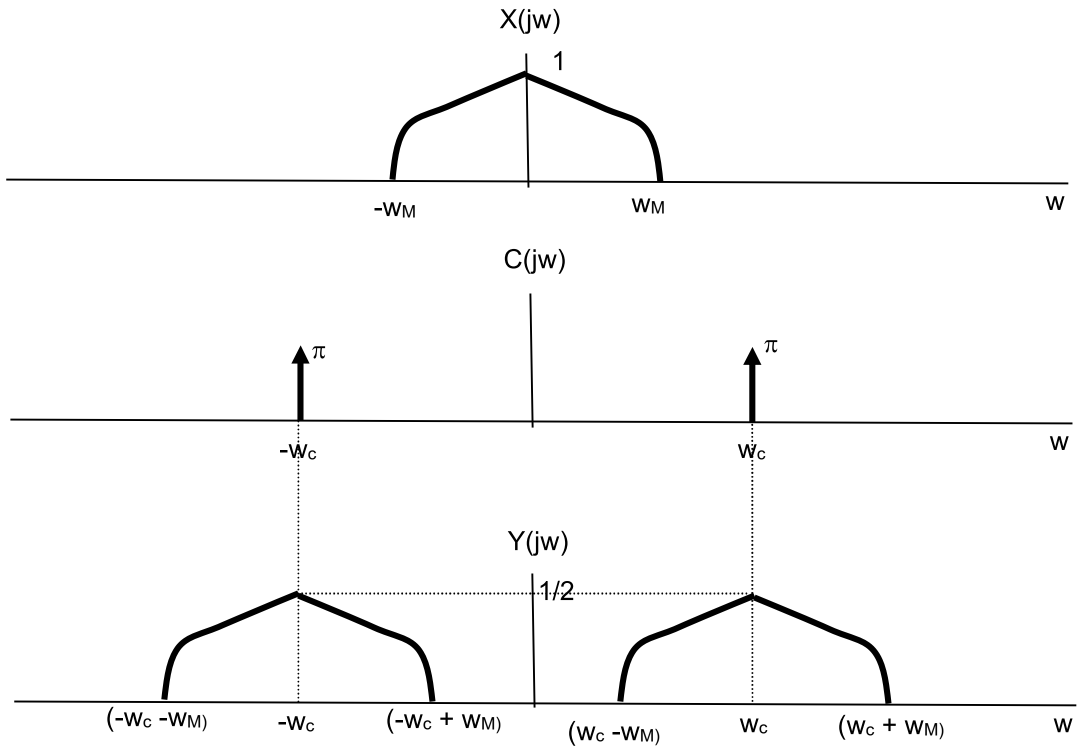
It would be equally effective to only use one of the two paths shown above to modulate and demodulate signals. And, designer commonly use the sinusoidal carrier signal {x(t)=cos(w_ct) to modulate the signal and simplify the communication system and has equally effective solutions.



Below are the derivations for the Fourier transform of the modulated signal Y(jw) when the sinusoidal carrier signal c(t) is used:

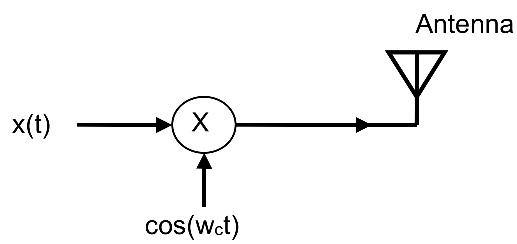
$$
\begin{aligned}
&Let\ c(t) = \cos(jw_c t) \therefore \\
&Given \rightarrow F\{c(t)\} = C(jw) = \pi\left[\delta(w - w_c) + \delta(w + w_c)\right] \\
&y(t) = x(t)\cos(jw_c t) \\
&Applying\ Fourier\ Transform\ multipication\ property \\
&Y(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)C(j(w - \theta)d\theta \\
&Applying\ the\ impluse\ to\ the\ Convolution\ \text{int}egration\ we\ get: \\
&Y(jw) = \frac{1}{2}\left[X(j(w - w_c) + X(j(w + w_c)\right]
\end{aligned}
$$

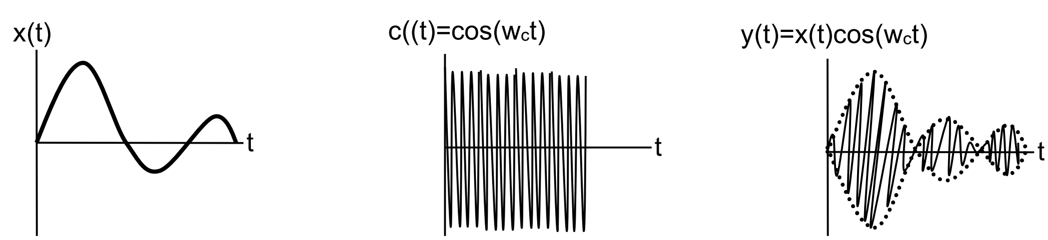
From the results, we can conclude that the modulated signal y(t) is information-bearing signal x(t) shifted by the carrier frequencies - w_c and $+w_c$ which is shown below:



When using sinusoidal Amplitude Modulation, it is required that $w_c > w_M$ (Carrier Freq > Max. Signal Freq.) in order for the information-bearing signal to be recoverable. As it can be seen from the diagram, if $w_c \leq w_M$ then there will be an overlap between two replicas which mean that the information-bearing signal x(t) is not recoverable from the modulated signal y(t).

The following diagram provides a time-domain representation of the Amplitude Modulation process:





a) Information-Bearing Signal, x(t)

b) Carrier Signal, c(t)

c) Modulated Signal, y(t)

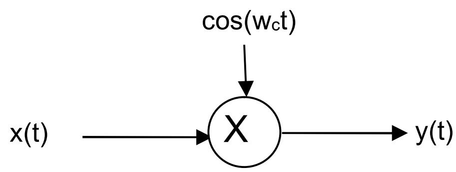
- Example – Available Modulator/Demodulator (Modems)

  Refer to MC 1496 Balanced Modulator/Demodulator data sheet for description of a typical FM/AM Modem available on the Market.

  MC 1496 is priced at less than $1 in 2010.

7.3. Sinusoidal Amplitude Demodulation - Synchronous and Asynchronous

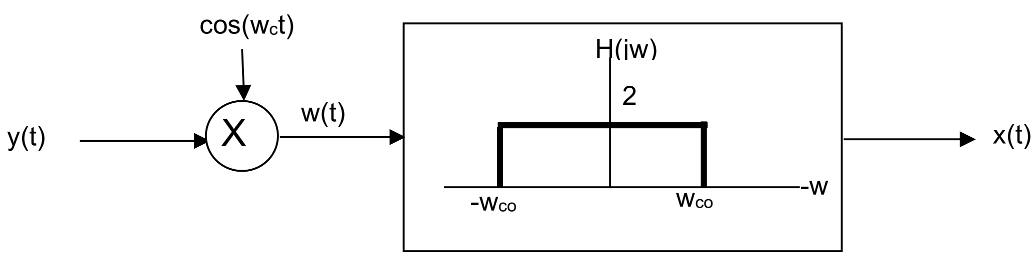
Pervious section introduced Amplitude Modulation (AM) for both complex exponential and sinusoidal carrier signal. As discussed earlier, the sinusoidal AM is simpler and equally effective which is the reason it is commonly used.



In this section, we will introduce synchronous and asynchronous demodulation for the amplitude modulated signal.

The synchronous demodulation requires that the demodulating carrier signal be synchronized with the modulating carrier signal which means there is no phase shift between the signals. Additionally, sinusoidal amplitude demodulation requires that the carrier frequency be higher than the maximum frequency of information-bearing signal ($w_c > w_M$).

Information-bearing signal x(t) is modulated using sinusoidal carrier signal cos(w_ct) which results in the modulation signal of y(t)=x(t)cos(w_ct). Synchronous demodulation is built on the fact that the information-bearing signal can be recovered by multiplying y(t) with the sinusoidal carrier signal (cos w_ct) and applying a low pass filter.



Low Pass Filter

Here are the derivations that prove we are able to use the above demodulation technique to recover the information-bearing signal x(t) from the modulated signal y(t):

$$y(t) = x(t)\cos w_c t \quad \text{modulated signal}$$

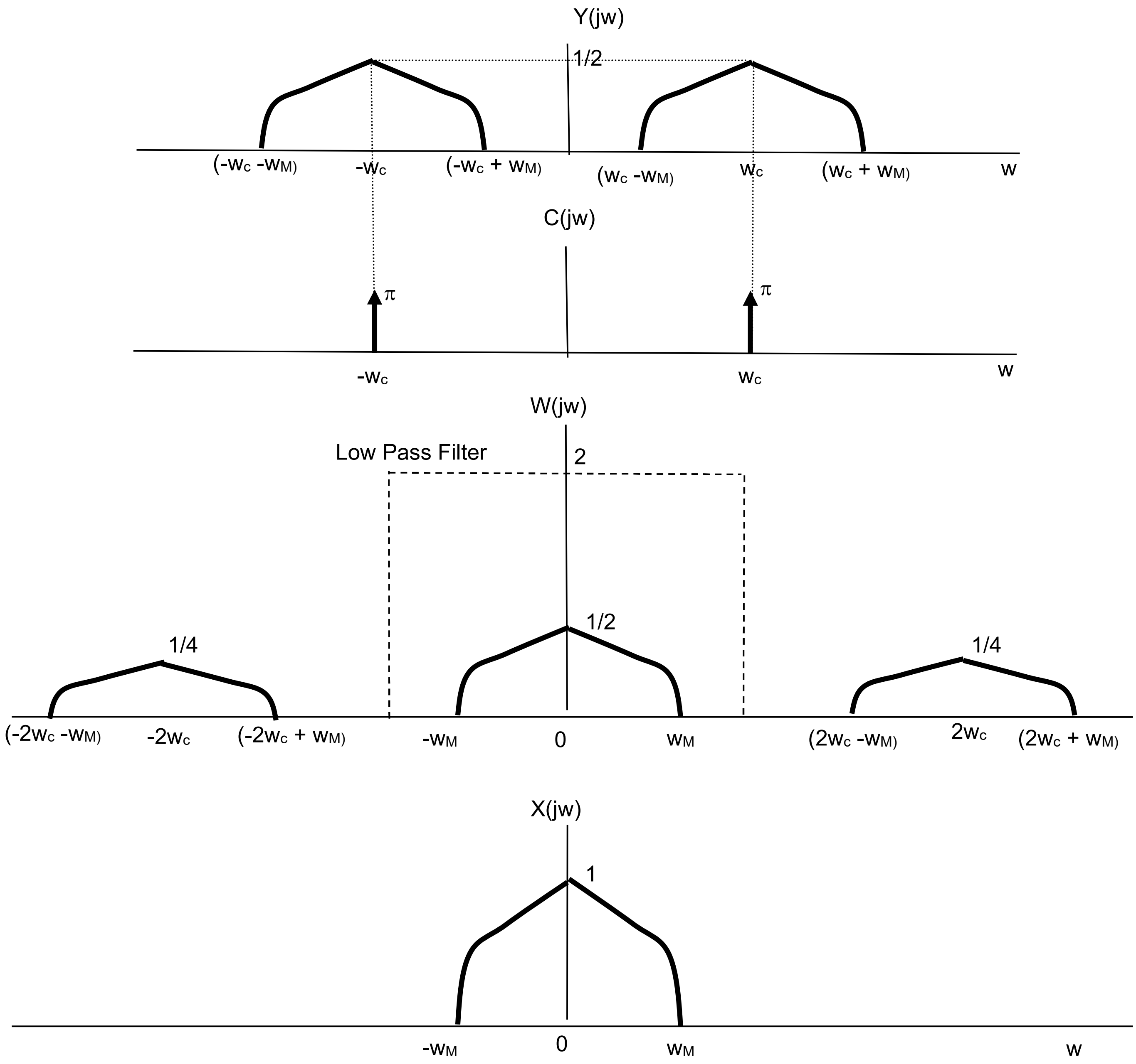
$$w(t) = y(t)\cos w_c t = x(t)\cos^2 w_c t \quad \text{intermidate function}$$

$$Given \rightarrow \quad 2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

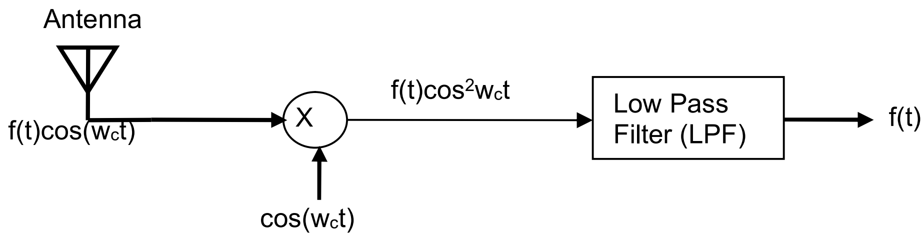
$$\cos^2 w_c = \frac{1}{2} + \frac{1}{2}\cos 2w_c t$$

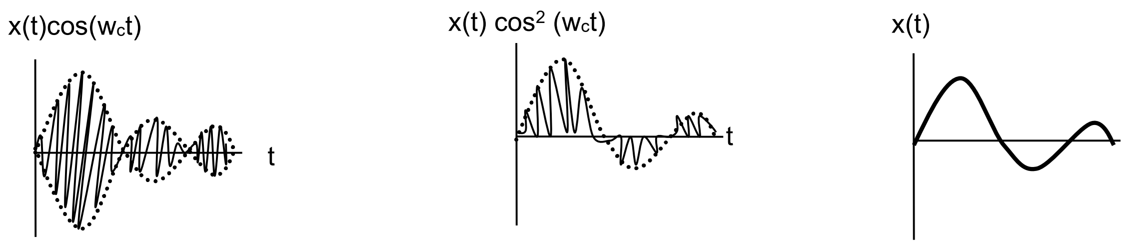
$$\therefore w(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2w_c t$$

We are able to extract x(t) by passing the intermediate function w(t) through a low pass filter with a gain of 2 and cut off frequency w_{c0} such that "$w_M < w_{c0} < (2w_c - w_M)$". Below are the frequency spectrum diagrams representing the steps of AM modulation and demodulation graphically:



The following diagram provides a time-domain representation of the demodulation process:

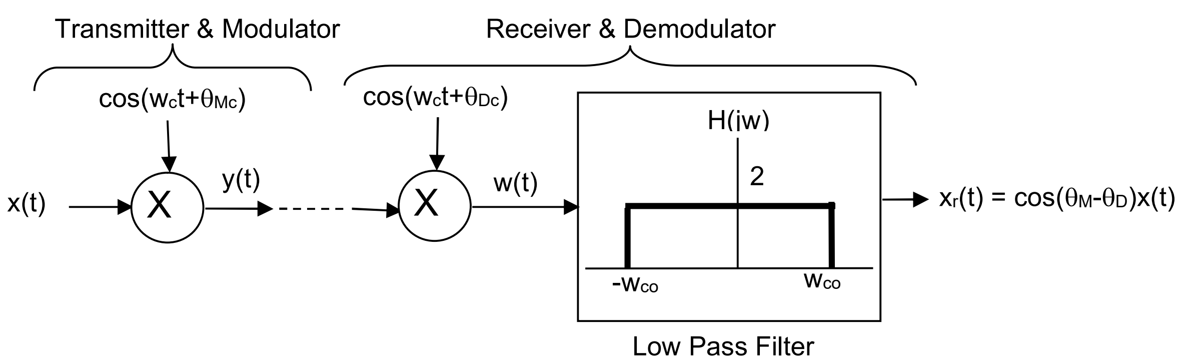




a) Modulated Signal, y(t)   b) Carrier Signal, c(t)   c) Input Signal, x(t)

The one challenge with synchronous demodulation is the requirement that the carrier signal at the receiver used to demodulate has to be synchronized with the signal used to modulate the signal at the transmitter (no phase shift). The asynchronous sinusoidal demodulation removes the requirement that the carrier signal at transmitting and receiving side have to have be in-phase (have the same phase).

Asynchronous sinusoidal demodulation occurs when the demodulating signal w(t) is not in-phase with modulating signal y(t)



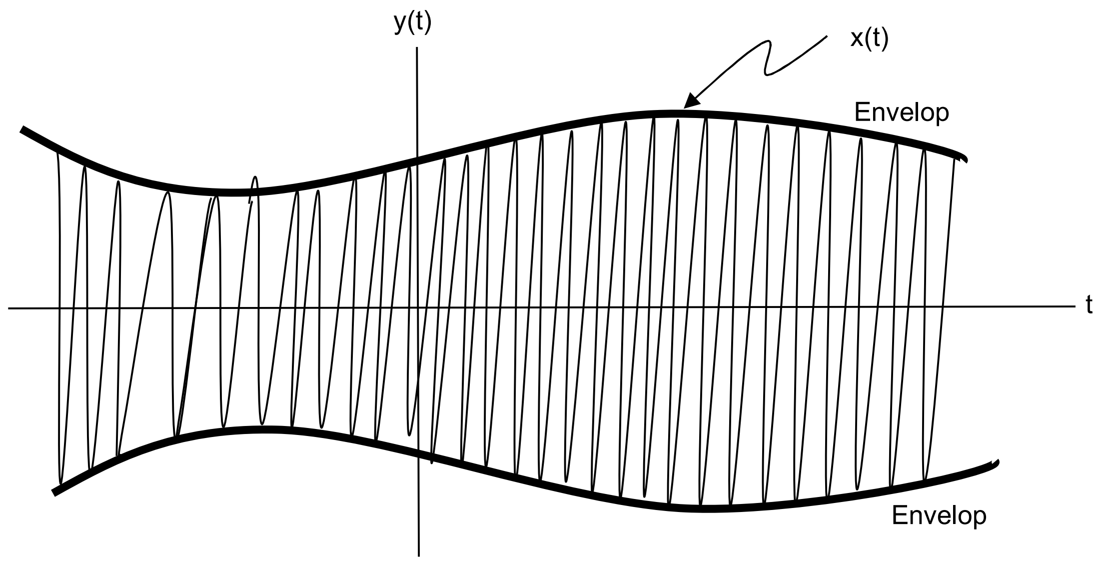
Where:

$$c(t) = \cos(w_c t + \theta_{Mc})$$
$$y(t) = \cos(w_c t + \theta_{Mc})x(t)$$
$$w(t) = \cos(w_c t + \theta_{Dc})y(t)$$
$$\therefore w(t) = \cos(w_c t + \theta_{Mc})\cos(w_c t + \theta_{D c})x(t)$$

Given trigonometry relation $\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$, w(t) can be rewritten as:

$$w(t) = \frac{1}{2}\cos(\theta_{Dc} - \theta_{Mc})x(t) + \frac{1}{2}\cos(2w_c t + \theta_{Dc} + \theta_{Mc})x(t)$$

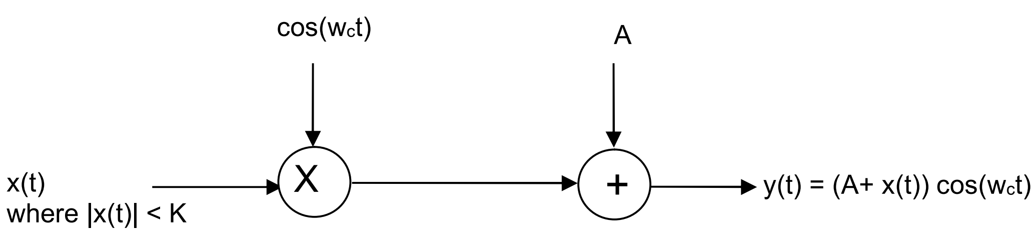
In this situation, we can attempt to recover information-bearing signal x(t) from the modulated signal y(t) by only using the envelop (Peaks):



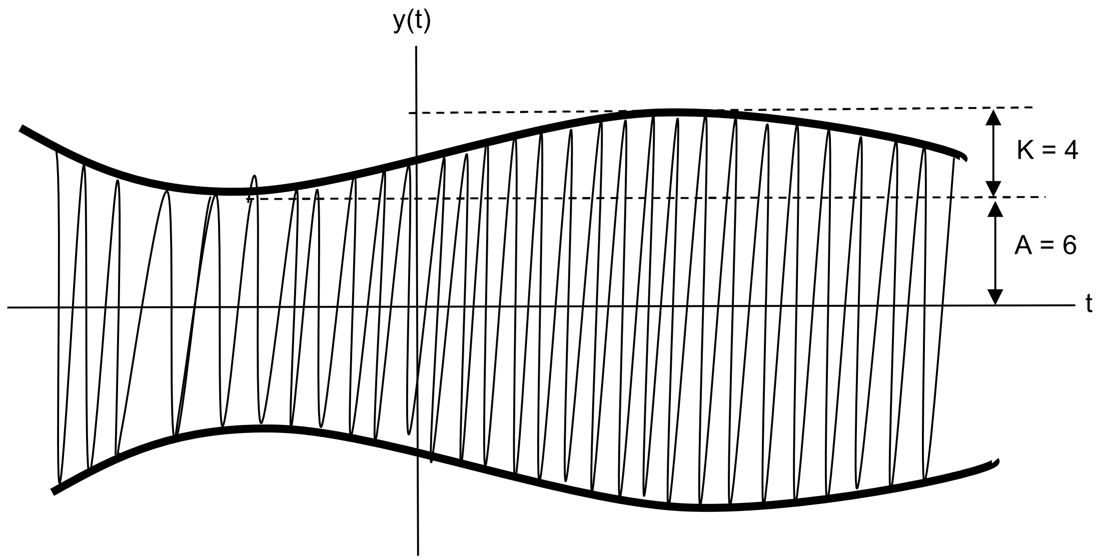
The two conditions required for this type of demodulation are:

1) x(t) must be positive
2) Carrier frequency w_c is much higher than the maximum signal frequency, w_M

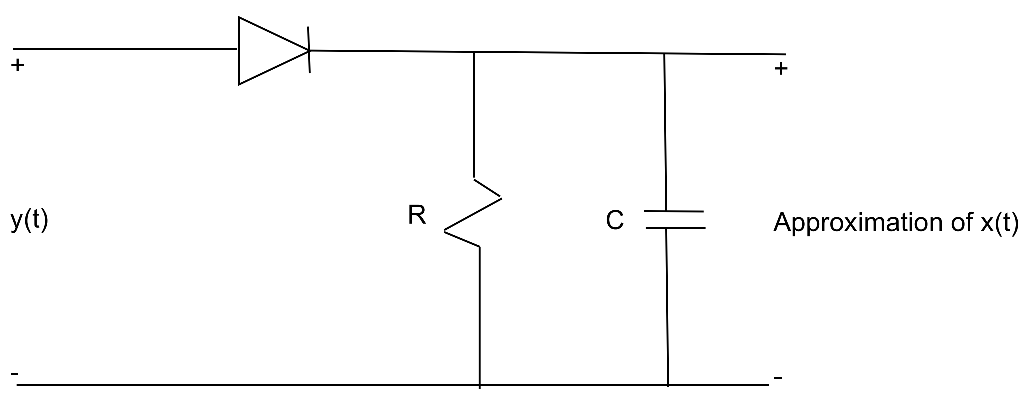
An example of modulator/demodulate system used to implement the asynchronous demodulation is outlined below:



The demodulation using envelop detector requires "A" to be sufficiently large so that x(t) + A is always positive. Since K is the maximum amplitude of x(t) then in order for x(t) + A to be positive, A must be larger than K (A >K). The ratio of K/A is called the modulation index m (m=K/A). You may also see m expressed in percent and called percent modulation. For example the modulated signal y(t) shown below has m =K/A=2/3= 67% modulation:



In order to derive a close approximation to the x(t), we can utilize the following low pass circuit:



Time Constant τ=1/RC must be significantly smaller than maximum x(t) period ($T_M = 2\pi/w_M$) and larger than carrier period ($T_c = 2\pi/w_c$)

$T_C << \tau << T_M$

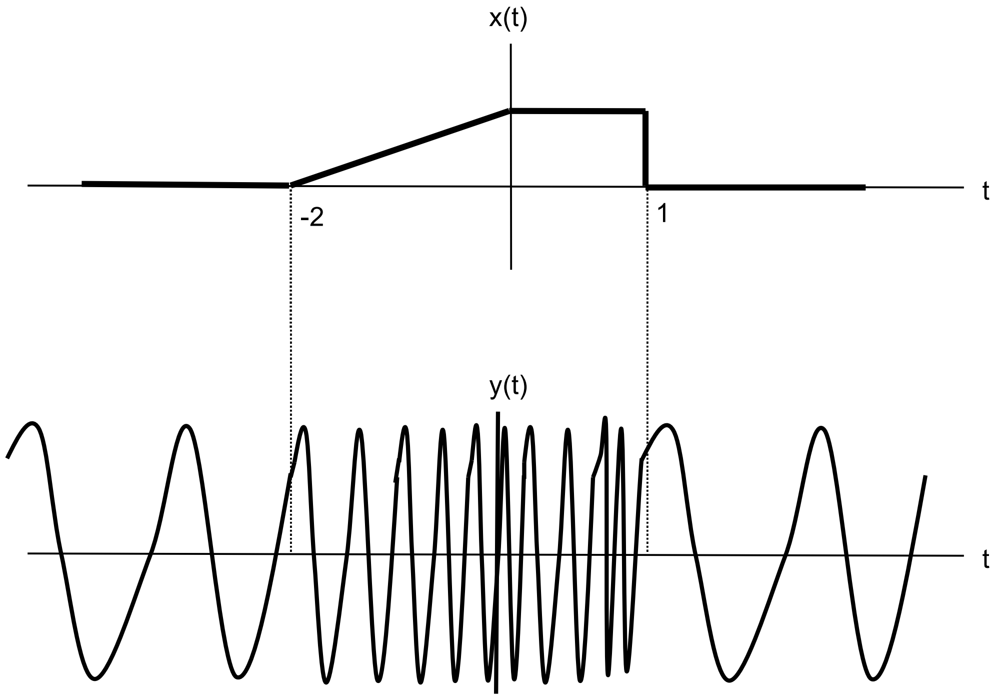
An advantage of asynchronous modulation-demodulation over synchronous approach is that it does not need synchronized carriers at the modulation and demodulation. Disadvantage is that it requires the transmitter to transmit higher power since x(t) must be positive and A>K. Although it adds cost to the transmitter, it reduces the cost of the receivers. For example in the case of an AM radio, this is a desirable trade off, since there is only one transmitter station but many receivers.

7.4. Sinusoidal Frequency Modulation (FM)

Frequency Modulation (FM) is another important modulation technique which is used in a variety of communication system including the FM Radio systems. In Frequency Modulation, the information-bearing or modulating signal x(t) controls the frequency of the carrier signal c(t).

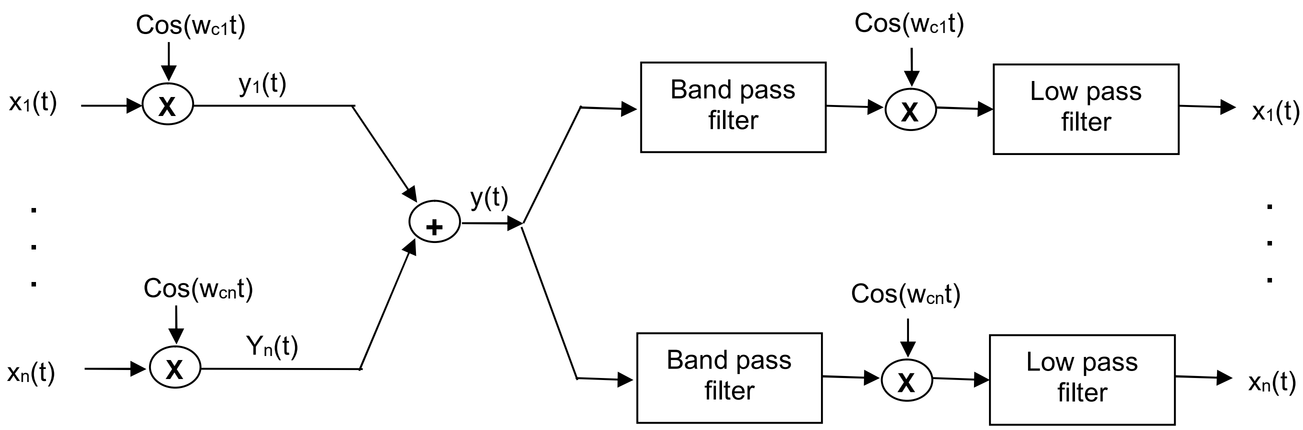
Advantage of FM over AM comes from the fact the transmitted signal has constant amplitude. This enables the transmitter to use maximum power all the time since the information is carried by varying the frequency of transmitted signal. This results in higher quality reproduction which is the reason most music radio station are FM. On the other hand FM required a larger range of frequency for the signal and it is a nonlinear signal which means our techniques are not sufficient to analyze FM systems.

Here is an example of Frequency Modulation where x(t) is the information-bearing signal and y(t) is the modulated signal.:

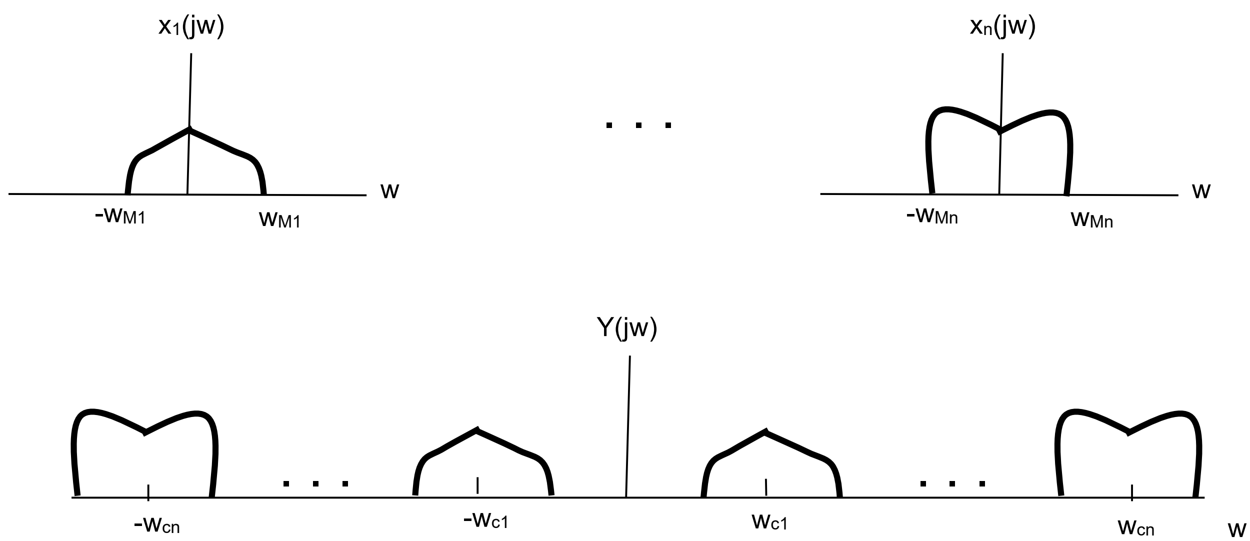


7.5. Frequency-Division and Time-Division Multiplexing

In addition to using the modulation (shifting signal frequency) for matching the optimal frequency range of the selected channel, the modulation/demodulation technique may be used to transmit many signals on the same channel. This is accomplished by allocating each signal different parts of the frequency spectrum. This technique is referred to as Frequency-Division Multiplexing (FDM). Below is a system diagram for modulation and demodulation required for Frequency-Division Multiplexing:



The key requirement is that the signal carrier frequencies be far enough from each other so that frequency spectrum signals do not overlap. Below is an example of n signals being Frequency-Division Multiplexed onto a single channel signal as shown by the following frequency spectrum of the signals:



Another way to share the channel is Time-Division Multiplexing (TDM) which means each signal is given a defined period of time to use the channel. In other word, the channel is shared over time and at any given time there is only one signal being transmitted through the channel.

7.6. Common Modulation Techniques

- Analog Modulation

  - AM
    Amplitude modulation (AM) is a technique used modulate analog signal by changing the amplitude of the carrier signal.
  - FM
    Frequency modulation (FM) is a technique used modulate analog signal by changing the frequency of the carrier signal.
  - PM
    Phase modulation (FM) is a technique used modulate analog signal by changing the phase of the carrier signal.

- Digital Modulation

  - ASK
    Amplitude-shift keying (ASK) represents digital data as changes in amplitude of carrier signal.
  - FSK
    Frequency-shift keying (FSK) represents digital data as changes in Frequency of carrier signal.
  - CPM
    Continuous phase modulation (CPM) is a form of digital data modulation.

- Digital Data Transmission Support Encodings

  - NRZI
    Non-Return-to-Zero-Inverted (NRZI) is a method of mapping a binary signal to a physical signal for transmission. The two level NRZI signal has a transition at a clock boundary if the bit being transmitted is a logical 1, and does not have a transition if the bit being transmitted is a logical 0.

7.7. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 8.

- Lathi, B. Modern Digital & Analog Communication Systems (1998) Oxford University Press
  Chapter 4.

- Stremler, F. Introduction to Communication Systems (1990) Addison-Wesley Publishing Company
  Chapt 5,6 and 7.

7.8. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 8. Laplace Transform

Key Concepts and Overview

- Laplace Transform
- Inverse Laplace Transform
- Region Of Convergence (ROC)
- Laplace Transform Properties
- Application of Laplace Transform to LTI Systems
- Additional Resources

8.1. Laplace Transform “X(s) = L{x(t)}”

The Continuous-Time Fourier Transform has proven to be an effective tool in analysis of LTI system behavior across a broad range of signals. Fourier Transform provides representation of signals as linear combinations of complex exponential signals in the form:

e^{st} where s=jw “pure imaginary”

This chapter introduces Laplace Transform which is an extension of continuous Fourier Transform. The Laplace Transform extends the definition of s such that:

s = σ + jw “s is a complex number”

In addition to covering a broader range of signals, Laplace Transforms enables analysis of system that may only be stable in a some regions. Later in this chapter, Laplace transform will be used in analysis of such LTI systems. Throughout this chapter, we will discuss bilateral Laplace transform. There is a special case Laplace transform which is called Unilateral Laplace transform where $t \leq 0^-$ is assume to be zero. Unilateral Laplace transform is typically used for LTI systems with zero initial conditions.

In this chapter, it will be demonstrated that there exist many similarities between Fourier transform and Laplace transform operation. Therefore we can leverage our knowledge of Fourier transform and take advantage of Laplace transform’s additional benefits and flexibility. The following relation is a restatement of result of earlier work on time-domain to frequency domain transforms:

$$y(t) = H(s)e^{st} \quad where \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where

- s=jw (pure imaginary)
- Frequency response H(s) is the Fourier Transform of impulse response h(t).

The above relationship may be generalized by stating that s is a complex variable (s=σ+jw). The change in the value of s to a complex numbers instead of pure imaginary yields:

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$ Referred to as the Laplace Transform equation (s=σ+jw)

Laplace Transform is Represented by $x(t) \xrightarrow{L} X(s)$ or L{x(t)} = X(s)

It can be seen from the following equation that when s=jw, Laplace transform changes to a Fourier transform:

$$X(jw) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = X(s)\big|_{s=jw}$$

Another observation is the relationship between the Fourier Transform and Laplace Transform when (s=σ+jw). The following derivation demonstrates the relationship:

$$X(S) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+jw)t}dt = \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-jwt}dt = F\{x(t)e^{-\sigma t}\}$$

or

L{x(t)} = F{x(t)e^{-at}}

- ❖ Example – Laplace Transform derivation
  - ➢ Example 1 – Find the Laplace transform of the signal x(t) = [2e^{-at} + 5e^{-bt}]u(t).

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty}\{2e^{-at} + 5e^{-bt}\}u(t)e^{-st}dt = \int_{0}^{\infty}\{2e^{-at} + 5e^{-bt}\}e^{-st}dt$$

$$X(s) = \int_{0}^{\infty}[2e^{-(a+s)t} + 5e^{-(b+s)t}]dt = \int_{0}^{\infty} 2e^{-(a+\sigma+\mathrm{jw})t}dt + \int_{0}^{\infty} 5e^{-(b+\sigma+\mathrm{jw})t}]dt$$

$$X\{s) = -\frac{2}{a+s}e^{-(a+s)t}\Big|_0^{\infty} - \frac{5}{b+s}e^{-(b+s)t}\Big|_0^{\infty}$$

$$X(s) = \frac{2}{a+s} + \frac{5}{b+s} = \frac{7s + (2a+5b)}{(a+s)(b+s)}$$

Another consideration is the convergence which requires the integrals to have values less than infinity:

$$X(s) = \int_{0}^{\infty} 2e^{-(a+\sigma+\mathrm{jw})t}dt + \int_{0}^{\infty} 5e^{-(b+\sigma+\mathrm{jw})t}]dt = \int_{0}^{\infty} 2e^{-(a+\sigma)\mathrm{t}t}e^{-jwt}dt + \int_{0}^{\infty} 5e^{-(b+\sigma)\mathrm{t}t}e^{-jwt}dt$$

$$\mathit{Convergence\ requires}:$$

$$a + \sigma > 0 \rightarrow \sigma > -a \quad \rightarrow \quad \mathrm{Re}\{s\} > -a$$

$$b + \sigma > 0 \rightarrow \sigma > -b \quad \rightarrow \quad \mathrm{Re}\{s\} > -a$$

Latter in this section the topic of Region of Convergence (ROC) will be covered in more detail.

  - ➢ Example 2 - Find the Laplace transform for the signal x(t) = 5 sin(3t) e^{-2t} u(t)

Solution:

Student Exercise

8.2. Inverse Laplace Transform "$x(t)=L^{-1}\{x(t)\}$"

A given transform is only useful if an inverse transform exists. Inverse of Laplace transform exists only if X(s) converges which will be discussed in detail in the next section. In this section, the focus is on developing the Inverse Laplace transform. Let's start with the relationship between the Fourier Transform and Laplace Transform when (s=σ+jw) from previous section:

$$X(S) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + jw)t}dt = \int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-jwt}dt = F\{x(t)e^{-\sigma t}\}$$

The following relationship may be derived by applying the inverse Fourier transform integral $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{jwt}dw$:

$$x(t)e^{-\sigma t} = F^{-1}\{X(\sigma + jw)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + jw)e^{jwt}dw$$

Multiply both sides of the above equation by $e^{\sigma t}$:

$$x(t) = \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty} X(\sigma + jw)e^{jwt}dw$$

or

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + jw)e^{(\sigma + jw)t}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}dw$$

Replace w with s= σ + jw which means the limits of the integral change to (σ + j∞) and (σ + j∞). finally dw is replaces with ds/j resulting in the Inverse Laplace transform equation:

. $\boxed{x(t) = \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds}$ Referred to as the inverse Laplace transform equation (s=σ+jw)

❖ Example – Inverse Laplace Transform

➢ Example 1 – Find the signal, x(t), with Laplace Transform $X(s) = \frac{1}{(s+2)(s+3)}$.

Solutions:

$$x(t) = \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds \quad \text{inverse Laplace transform equation}$$

$$x(t) = \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{1}{(s+2)(s+3)}e^{st}ds$$

Expand the value of X(s)

$$\frac{1}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} = \frac{(A+B)s + (3A+2B)}{(s+2)(s+3)}$$

In order for the two side to be equal:

A+B = 0

3A+2B =1

Solve the system of two equation and two unknowns to find A=1 and B =-1, therefore:

$$x(t)=\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\{\frac{1}{(s+2)}-\frac{1}{(s+3)}\}e^{st}ds=\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+2)}e^{st}ds-\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+3)}e^{st}ds$$

Need to make the power of e^{st} and the denominator the same by multiply/dividing powers of e:

$$x(t)=\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+2)}\frac{e^{2t}}{e^{2t}}e^{st}ds-\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+3)}\frac{e^{3t}}{e^{3t}}e^{st}ds$$

$$x(t)=\frac{e^{-2t}}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+2)}e^{(s+2)t}ds-\frac{e^{-3t}}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{1}{(s+3)}e^{(s+3)t}ds$$

$$x(t)=\frac{e^{-2t}}{j2\pi}e^{(s+2)t}\Big|_{\sigma-j\infty}^{\sigma+j\infty}-\frac{e^{-3t}}{j2\pi}e^{(s+3)t}\Big|_{\sigma-j\infty}^{\sigma+j\infty}$$

$STEPSMISSING$

$$x(t)=\{e^{-2t}-e^{-3t}\}u(t)\quad for\quad \sigma=\mathrm{Re}\{s\}>-2$$

As it can been seen from the above example, we can find the inverse Laplace transform with the use of inverse Laplace transforms equation for a class of ration Laplace transforms of the form:

$$X(s)=\sum_{k=1}^{N}\frac{A_k}{s+a_k}\quad where\ Ak\ and\ ak\ are\ cons\mathrm{tan}ts$$

$x(t)=\sum_{k=1}^{N}A_k e^{-a_k t}u(t)$ if ROC is Re{s} >-a_k

$x(t)=\sum_{k=1}^{N}-A_k e^{-a_k t}u(-t)$ if ROC is Re {s} <-a_k

- Example 2 – Find the signal x(t) with Laplace transform $X(s)=\dfrac{2}{s^2+12s+32}$

Solution:

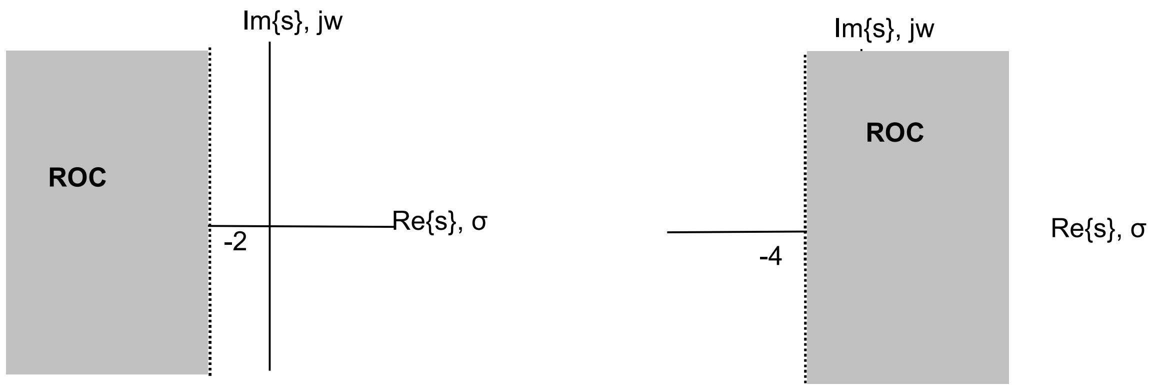
Student Exercise

8.3. Region Of Convergence (ROC)

The Laplace Transform Region of Convergence (ROC) is the range of value of (s=σ + jw) where valid Laplace transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ exists. For valid Laplace transform to exist, we only need to show that the following inequality is true:

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$$

Typically, ROC is shown in a rectangular coordinate which is referred to as s-plane. The vertical axis (jw-axis) is the imaginary part of s (Im{s}=w) and horizontal axis (σ-axis)is the real part of s (Re{s}= σ). The shaded area of the s-plane is the ROC. Below are two examples of ROC in s-planes:



a) ROC is Re{s} < -2

b) ROC is Re{s} > -4

In the case where the signal x(t) is a linear combination of real or complex exponential, its Laplace transform is rational. Which means, it is ratio of numerator over denominator in term of complex variable s:

$$X(s) = \frac{N(s)}{D(s)}$$

As simple example would be x(t) = $5e^{-4t}u(t) + 2e^{-3t}u(t)$. In this case x(t) is a linear combination of real exponential. Its Laplace transform is rational and can be calculated by:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} \{5e^{-4t} + 2e^{-3t}\}e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} [5e^{-(4+s)t} + 2e^{-(3+s)t}]dt = -\frac{2}{4+s}e^{-(3+s)t}\Big|_0^{\infty} - \frac{5}{3+s}e^{-(3+s)t}\Big|_0^{\infty}$$

$$X(s) = \frac{2}{4+s} + \frac{5}{3+s}$$

Applying the convergence condition $\int_{-\infty}^{\infty} | x(t) | e^{-\sigma t} dt < \infty$ and s=σ+jw to this function yields:

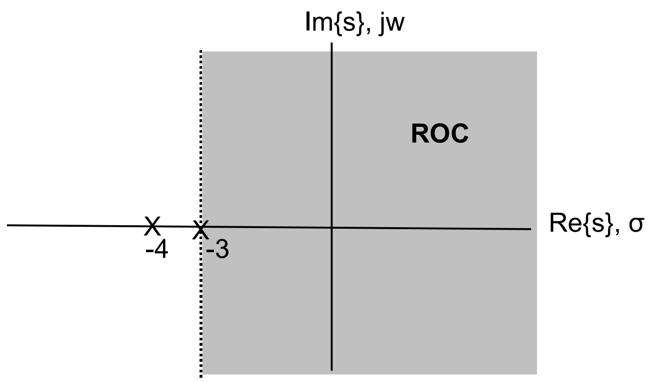
$$\int_{-\infty}^{\infty} | x(t) | e^{-st} dt = \int_{0}^{\infty} [5e^{-(4+s)t} + 2e^{-(3+s)t}]dt$$

Note that above quantity is only less than infinity if the exponentials are decaying which means powers of e must be less than 0, therefore:

Re {4+ s} > 0 & Re{3 + s} > 0 → Re{s} > -4 and Re{s} > -3 → ROC is Re{s} > -3

Note that we could have determine the ROC by ensuring that the denominators in the equation $X(s) = \frac{2}{4+s} + \frac{5}{3+s}$ are all less than 0.

The roots of the numerators are called zeros and are shown by "o" on the s-plane. The roots of the denominator are called poles and are shown by "x" on the s-plane. Additionally, poles are where X(s) is infinity and zeros are where X(s) is zero. ROC and location of poles are related, therefore the zero-pole plot is a useful tool in identifying ROC. Below is the s-plot or zero-pole plot for X(s):



ROC is Re{s} > -3

We can generalize the process of determining the ROC for all rational Laplace transforms $X(s) = \frac{N(s)}{D(s)}$ by considering the following:

- If x(t) is a linear combination of real or complex exponentials then the Laplace transform is rational.
- if the power of s in the numerator is higher than the power of s in the denominator, X(s) becomes unbounded as s approaches infinity.
- If the power of numerator is less than denominator, poles identify the ROC.

In general, ROC for rational Laplace transform can be defined by the following relationships:

$$X(s) = \frac{N(s)}{D(s)} = \sum_i \frac{A_i}{s + a_i}$$ ➔ ROC is where:

Re{s} > - Re{a_i} when $A_i > 0$
Re{s} < - Re{a_i} when $A_i < 0$

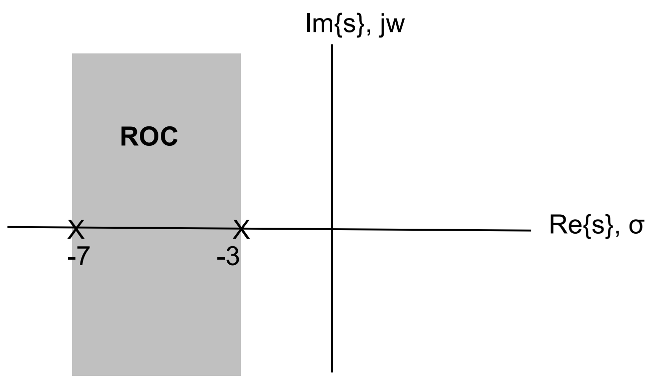
- Example – ROC for Rational Laplace Transform
  - Example 1 - Find the Laplace transform with its ROC for the function $x(t) = e^{-7t}u(t) + e^{-3t}u(-t)$.

    Solution:

    It is understood that L{x(t)}=X(s)= $\frac{1}{s+7} + \frac{-1}{s+3}$

    ROC for first term → R{s} > -7
    ROC for second term → R{s} < -3

    Therefore the ROC for X(s) is → -7 < R{s} < -3
    The following s-plan shows the ROC (shaded area):



ROC is Re{s} > -3

  - Example 1 – Find the Laplace transform with its ROC for the function $x(t) = [e^{-4t} + e^{-2t}\cos(5t)]\, u(t)$.

    Solutions:
    $x(t) = [e^{-4t} + \frac{1}{2} e^{-2t}\{ e^{j5t} + e^{-j5t} \}]\, u(t) = [e^{-4t} + \frac{1}{2}\, e^{(-2+j5)t} + \frac{1}{2}\, e^{(-2-j5)t}]u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} \{e^{-4t} + \frac{1}{2}e^{-(+2-j5)t} + \frac{1}{2}e^{-(+2+j5)t}\}e^{-st}dt$$

$$X(s) = \frac{1}{s+4} + \frac{1/2}{s+(2-j5)} + \frac{1/2}{s+(2+j5)}$$

$$\frac{1}{s+4} \Rightarrow ROC \quad \mathrm{Re}\{s\} > -4$$

$$\frac{1/2}{s+2-j5} \Rightarrow ROC \quad \mathrm{Re}\{s\} > -\mathrm{Re}\{2-j5\} = -2$$

$$\frac{1/2}{s+2+j5} \Rightarrow ROC \quad \mathrm{Re}\{s\} > -\mathrm{Re}\{2+j5\} = -2$$

L{[e^{-4t} + e^{-2t} cos(5t)] u(t)} = $\frac{1}{s+4}+\frac{1/2}{s+(2-j5)}+\frac{1/2}{s+(2+j5)} \Rightarrow ROC \quad \mathrm{Re}\{s\} > -2$

- Example 3 – Find the Laplace transform with its ROC for the function
  x(t) =

  Solution:

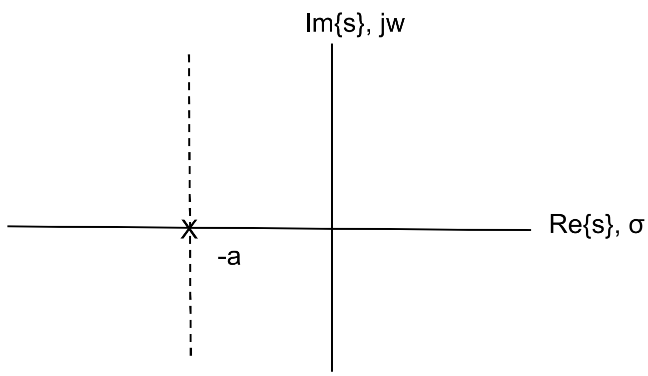
  Student Exercise

Now that we have experienced the process of finding ROC for rational Laplace transform, it is time to expand the discussion to the process of identifying ROC from the algebraic X(S) in frequency domain and characteristic of x(t) in time domain. The ROC are values of s for which |x(t|$e^{-\sigma t}$ is integratable:

$$\int_{-\infty}^{\infty} | x(t) | e^{-\sigma t} dt < \infty$$ where σ = Re{s} = RE{σ+jw}

From the above requirement and the fact that ROC is shown on s-plane, we can derive 8 rules or properties to guide the process of ROC determination. The remainder of this section outlines the these 8 rules:

Rule 1 ROC of X(s) is defined by parallel lines to the jw-axis in the s-plane since the ROC is define only by the real part of s (or σ) as shown below:



Rule 2 ROC of rational Laplace transforms does not include poles.
Poles are roots of denominator of X(s) where X(s) → ∞. Therefore X(s) ROC cannot contain poles.

Rule 3 ROC covers all of s-plane when x(t) has a finite duration and is absolutely integrateable.
It can be shown that for the above condition, the following relationship is true for all s:

$$\int_{-\infty}^{\infty} | x(t) | e^{-\sigma t} dt < \infty$$ "Meaning X(s) converges for all values of s"

Rule 4 All values of s where Re{s} > σ_0 is in the ROC if x(t) is right sided and the line Re{s} = σ_0 is in the ROC.

A right sided signal is one where has the form x(t)u(t-τ). The fact that Re{s} = σ_0 is in the ROC leads to the following relationship:

$$\int_{-\infty}^{\infty} | x(t) | e^{-\sigma_0 t} dt < \infty$$

The above inequality holds true for any Re{s} = $\sigma_1 > \sigma_0$

$$\int_{-\infty}^{\infty} | x(t) | e^{-\sigma_1 t} dt < \infty$$

From this inequality, we can conclude that for all values of s where Re{s} > σ_0 is in the ROC.

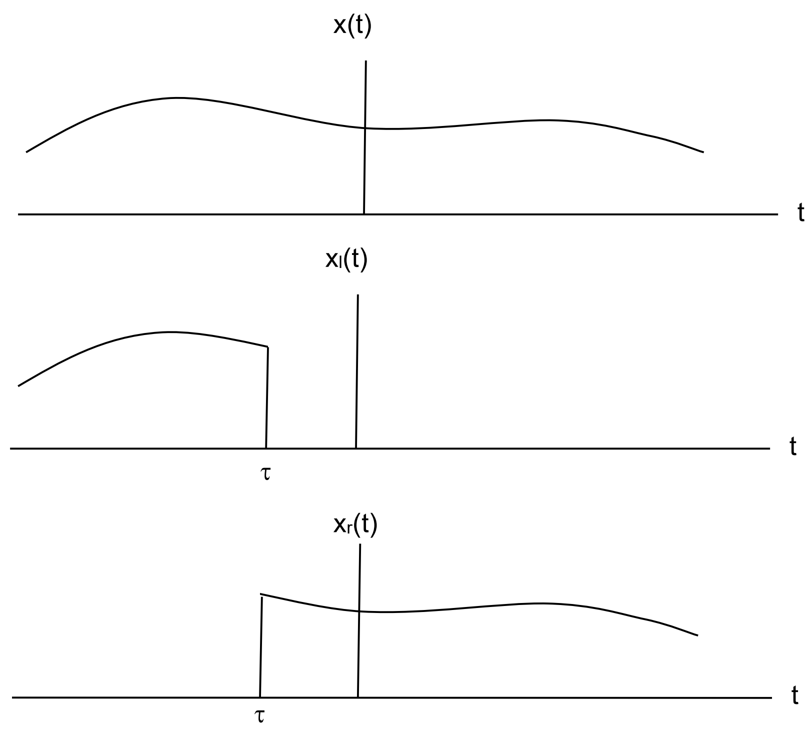
Rule 5 All values of s where Re{s} < σ_0 is in the ROC if x(t) is left sided {x(t)u(-t-τ)} and the line Re{s} = σ_0 is in the ROC.

The proof of this rule is similar to Rule 4 and students are encouraged to do the proof.

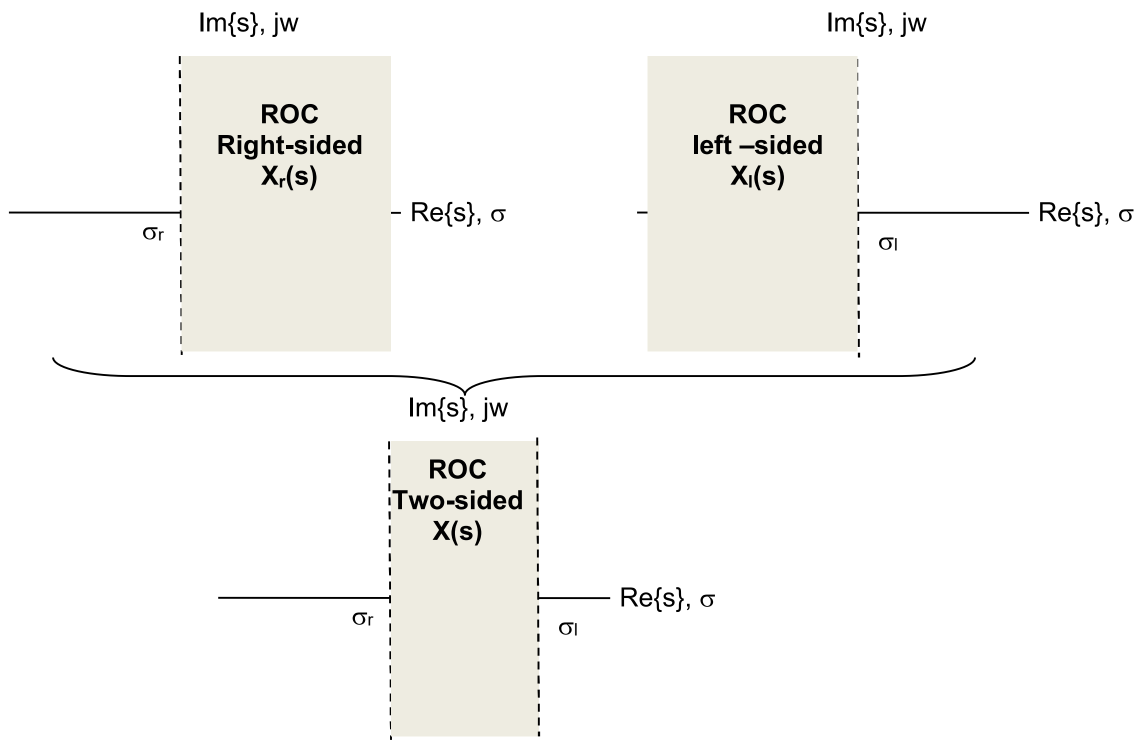
Rule 6 ROC includes a strip in the s-plane that contains line Re{s} = σ_0 when x(t) is two sided and the line Re{s} = σ_0 is in the ROC.

A two sided signal goes from –∞ to +∞ and we can leverage rules 4 and 5 in the proof of this rule.

x(t) can be restated as a sum of a right sided signal $x_r(t)$ and a left sided signal $x_l(t)$ as shown below:



Based on rules 4 and 5, we are able to draw the ROC for $X_l(s)$ and $X_r(s)$ as shown below:



<u>Rule 7</u> When the Laplace transforms X(s) of x(t) is rational, The following holds true:

1) ROC is bounded by poles or extends to infinity, and
2) no X(s) poles are constrained in the ROC.

<u>Rule 8a</u> When the Laplace transform X(s) of x(t) is rational and x(t) is right sided, its ROC is the region in the s-plane to the right of the rightmost pole.

<u>Rule 8b</u> When the Laplace transform X(s) of x(t) is rational and x(t) is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

8.4. Laplace Transform Properties

Laplace Transform has properties that are similar to the Fourier Transform. The following table contains the most common properties of Laplace Transform:

Property	**Time-Domain** **$x(t)$, $x_1(t)$, $x_2(t)$**	**Laplace s-Domain** **$X(s)$, $X_1(s)$, $X_2(s)$**	**ROC** **R, R_1, R_2**
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-t_0} X(s)$	R
Shifting in the s-Domain	$e^{s_0 t} x(t)$	$X(s - s_0)$	Shifted R *"s is in R if $(s - s_0)$ is in R"*
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(\frac{s}{a})$	Scaled R *"s is in R if s/a is in R"*
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Conjunction	$x^*(t)$	$X^*(s^*)$	R
Differentiation in t	$\frac{d^n x(t)}{dt^n}$	$s^n X(s)$	At least R
Differentiation in s	$(-1)^n t^n x(t)$	$\frac{d^n X(s)}{ds^n}$	R
Integration in t	$\int_{-\infty}^{t} x(\tau) d(\tau)$	$\frac{1}{s} X(s)$	At Least $R \cap [Re\{s\} > 0]$

Note:

∩ "Intersection of sets" resulting set includes only the common elements of the two sets.
U "Union of sets" resulting set includes all the elements of both sets.

One may proof these properties by apply the Laplace transform or Laplace inverse transform equations.

The following table contains the Laplace Transform of common signals and corresponding Regions of Convergence (ROC):

Function	Time Domain, x(t)	Laplace s-domain, X(s)	ROC
Unit Impulse	$\delta(t)$	1	All s
Ideal Delay	$\delta(t - T)$	e^{-Ts}	All s
Unit Step	u(t)	$\frac{1}{s}$	Re{s} > 0
Unit Step	-u(-t)	$\frac{1}{s}$	Re{s} < 0
Ramp	t u(t)	$\frac{1}{s^2}$	Re{s} > 0
n^{th} Power	$\frac{t^n}{n!}u(t)$	$\frac{1}{s^{n+1}}$	Re{s} > 0
n^{th} Power	$-\frac{t^n}{n!}u(-t)$	$\frac{1}{s^{n+1}}$	Re{s} < 0
Delayed Step	u(t- T)	$\frac{e^{-sT}}{s}$	Re{s} > 0
Exponential decay	$e^{-at}u(t)$	$\frac{1}{s+a}$	Re{s} > -a
Exponential decay	$-e^{-at}u(t)$	$\frac{1}{s+a}$	Re{s} < -a
n^{th} power exponential decay	$\frac{t^n}{n!}e^{-at}u(t)$	$\frac{1}{(s+a)^{n+1}}$	Re{s} > -a
n^{th} power exponential decay	$-\frac{t^n}{n!}e^{-at}u(-t)$	$\frac{1}{(s+a)^{n+1}}$	Re{s} < -a
Sine	$\cos(w_0 t)\, u(t)$	$\frac{s}{s^2+w_0^2}$	Re{s} < 0
Cosine	$\sin(w_0 t)\, u(t)$	$\frac{w_0}{s^2+w_0^2}$	Re{s} > 0
Exponential decaying cosine	$e^{-at}\cos(w_0 t)\, u(t)$	$\frac{s+a}{(s+a)^2+w_0^2}$	Re{s} > -a
Exponential decaying cosine	$e^{-at}\sin(w_0 t)\, u(t)$	$\frac{w_0}{(s+a)^2+w_0^2}$	Re{s} > -a
n^{th} order differentiation	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
n^{th} order step convolution	$u_{-n}(t) = u(t)*u(t)*\ldots$ Convolution of n u(t)	$\frac{1}{s^n}$	Re{s} > 0

8.5. Application of Laplace Transform to LTI Systems

Laplace transform is used in analysis of LTI system by relating the system input and output as shown below:

$$X(s)=L\{x(t)\} \longrightarrow \boxed{H(s) = L\{h(t)\}} \longrightarrow \begin{aligned} Y(s) &= L\{y(t)\} \\ &= L\{h(t)*x(t)\} = H(s)X(s) \end{aligned}$$

When H(s) has a ROC which includes the imaginary axis s=jw, then H(s) is the frequency response of the LTI system for s=jw. In general H(s) is referred to as the system function or transfer function. H(s) can be used to determine some key properties of LTI systems. Remainder of this section relates H(s) to system stability and causality.

As it was discussed earlier, a system with impulse response h(t) is causal when h(t) =0 for t < 0. This means a causal system is right sided. The fact the h(t) is right sided means that system function H(s) of a causal system has a ROC on the right half of s-plane. The converse of this statement is only true if H(s) is rational. This information can be used to determine that a system is causal by checking to see if its H(s) ROC is in the right plane.

- Example – Using Transfer Function H(s) to determine system causality
  - Example 1 - Determine if the system with transfer function $H(s) = \dfrac{1}{s+6}$ is causal.

    Solution:

    ROC for H(s) Re(s) > -6
    Since H(s) is rational and ROC is right sided therefore system is causal

  - Example 2 - Determine if the system with transfer function $H(s) = \dfrac{-9}{(s-5)(s+4)}$ is causal.

    Solution:

    rewrite → $H(s) = \dfrac{-1}{s-5} + \dfrac{1}{s+4}$
    ROC → -4 < Re{s} < +5

    Since H(s) is rational but ROC is not right sided therefore the system is NOT causal

Another major LTI system property is stability which can also be determined by examining the systems transfer function H(s). Let's start by remembering that a LTI system is stable when its impulse response is absolutely integrateable:

$$\int_{-\infty}^{\infty} | h(t) | \, dt < \infty$$

As discussed earlier a LTI system is stable if Fourier Transform converges. We know that Laplace Transform is reduced to Fourier Transform when s=jw therefore "A LTI system is stable if and only if transfer function H(S) ROC includes s=jw (Re{s}=0)"

- Example – Using Transfer Function H(s) to determine system stability

- Example 1 - Determine if the system with transfer function $H(s) = \frac{1}{s-2}$ is stable.

  Solution:

  ROC for H(s) → Re(s) > 2
  Since ROC does not include s=jw or Re{s}=0, the system is not stable.

- Example 2 - Determine if the system with transfer function $H(s) = \frac{-9}{(s-5)(s+4)}$ is stable.

  Solution:

  rewrite → $H(s) = \frac{-1}{s-5} + \frac{1}{s+4}$
  ROC → -4 < Re{s} < +5

  Since ROC contain s=jw or Re{s}=0, the system is stable.

8.6. Additional Resources

- ❖ Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter 9.

- ❖ Nilsson, J. Electrical Circuits. (2004) Pearson.
  Chapter 12 and 13.

8.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Chapter 9. Z-Transform

Key Concepts and Overview

- Z-Transform
- Inverse Z-Transform
- Region Of Convergence (ROC)
- Z-Transform Properties
- Application of Z-Transform in LTI Systems
- Additional Resources

9.1. Z-Transform, "X(z) = Z{x[n]}"

Z-transform is an extension of Discrete-Time Fourier Transform much like Laplace Transform was an extension of Continuous-Time Fourier Transform. In addition to covering a broader range of signal, Z-Transforms allows for analysis of Discrete-Time systems that are only stable in a specific region. Later in this chapter, Z-transform will be used in analysis of such LTI systems. Throughout this chapter, we will discuss bilateral Z transform. There is special case version of Z-transform which is called unilateral Z-transform where n $\leq 0^-$ is assumed to be zero. Unilateral Z-transform is typically used for LTI systems with zero initial conditions.

Further, we will show that there are many similarities between Fourier transform and Z-transform operation. Therefore we can leverage our knowledge of Fourier transform and take advantage of Z-transform's additional benefits and flexibility.

In developing the Z-transform, let's start with Discrete-Time Fourier transform from earlier chapters:

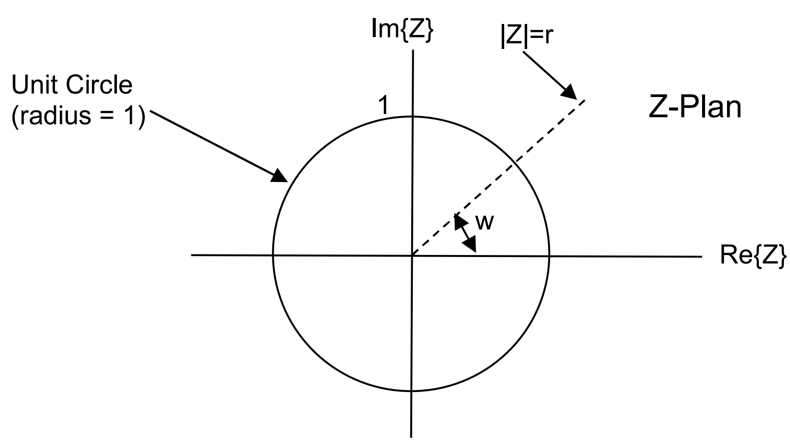
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

By setting Z = e^{jw}, the above relationship may be rewritten as the bilateral Z-transform where |Z| = |e^{jw}| = 1:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

In the general form of Z-transform, Z is expressed in polar form:

$z = re^{jw}$ where r is the magnitude and w is the phase angle of Z.



We can rewrite the Z-transform equation as:

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-jwn}$$

It can be rewritten in term of Fourier Transform:

$$X(re^{jw}) = F\{x[n]r^{-n}\}$$

Z-transform can be characterized based on value of "r" as shown below:

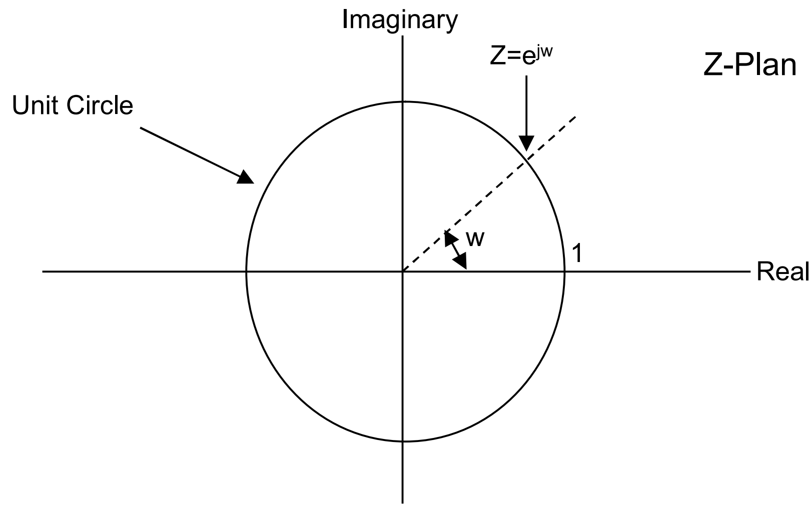
- Decaying when r>1
- Growing when r<1
- For r=1 → z-transform is reduced to Fourier transform $X(z)\big|_{z=e^{jw}} = X(e^{jw}) = F\{x[n]\}$

9.2. Inverse Z-Transform, "$x[n] = Z^{-1}\{X(z)\}$"

TBC

9.3. Region Of Convergence (ROC)

Z-transform Region Of Convergence (ROC) is define by using the unit circuit introduced earlier:



For X(z) to converge, Fourier transform of $x[n]r^{-n}$ must converge and ROC on Z-Plan are value of z for which X(z) converges. If ROC contains the circle then Fourier transform of x[n] also converges.

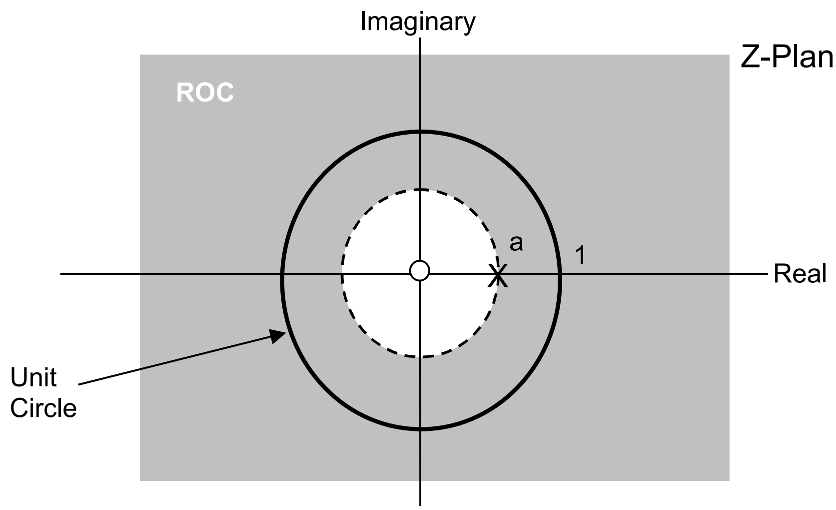
- Example 1. Find ROC for $x[n]=a^n u[n]$
  Apply the Z-transform analysis equation

  $$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n]z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

  For X(z) to converge → we require $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$ which means $|az^{-1}| < 1$ or |z|>|a|

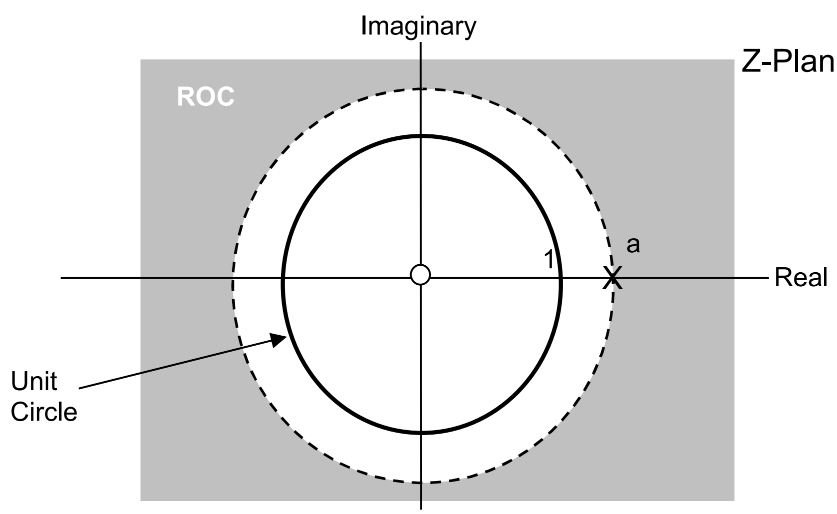
  Applying the finite sum we get: $X(z) = \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$ when |z|>|a|

  The requirement |z|>|a| defines the Region of Convergence (ROC) for X(Z) as the shaded area in the following pole-zero plot when 0<|a|<1:



Note:
"X" denotes Poles {denominator of X(z) = 0}
"O" denotes Zeros {Numerator of X(z) = 0)
This diagram is called "Pole-Zero Plot"

- If |a|>1 the ROC is shown below and note that now the unit circle is not included n ROC which means Fourier transform x[n] does not converge.



- Example 2. Find Region of convergence and z-transform for signal

$$x(n) = 7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left\{7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]\right\} z^{-n}$$

$$X(z) = 7\sum_{n=-\infty}^{\infty} (\frac{1}{3})^n u[n]z^{-n} - 6\sum_{n=-\infty}^{\infty} (\frac{1}{2})^n u[n]z^{-n}$$

$$X(z) = 7\sum_{n=0}^{\infty} (\frac{1}{3}z^{-1})^n - 6\sum_{n=0}^{\infty} (\frac{1}{2}z^{-1})^n$$

First term requires $|(1/3)z^{-1}|<1 \rightarrow |z|>1/3$
second term requires $|(1/2)z^{-1}|<1 \rightarrow |z|>1/2$

Therefore X(z) converges when |z|>1/2 and applying infinite sum relationship we get z-transform:

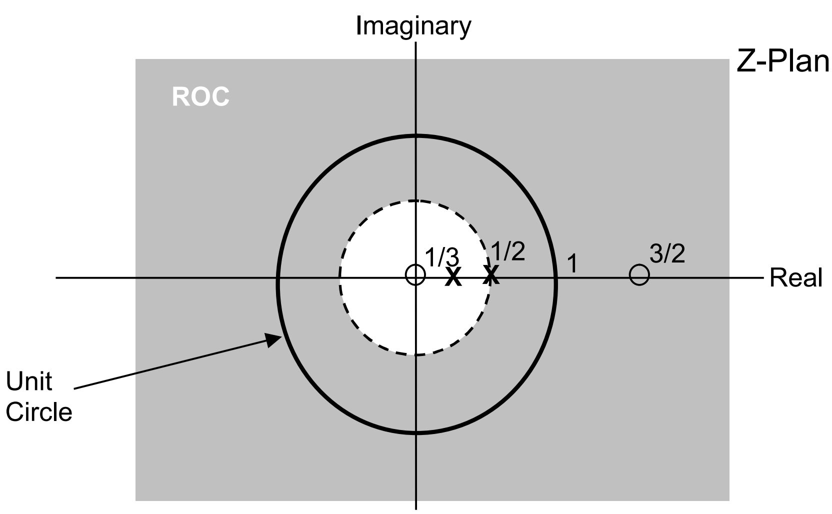
$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \; where\, |z| > 1/2$$

combining the fraction we get a rational function:

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})} \; where\, |z| > 1/2$$

Two Zeros: 0 & 3/2
Two poles: 1/3 & 1/2

- Pole-zero plot and ROC are shown below:



- Example 4. Find Region of convergence and z-transform for sequence

$$x(n) = (\frac{1}{3})^n \sin(\frac{\pi}{4}n)u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left\{ (\frac{1}{3})^n \sin(\frac{\pi}{4} n) u[n] \right\} z^{-n}$$

Apply Euler's Identity

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ (\frac{1}{3})^n \frac{1}{2j} \left[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n} \right] u[n] \right\} z^{-n}$$

$$X(z) = \frac{1}{2j} \sum_{n=0}^{\infty} (\frac{1}{3} e^{j\frac{\pi}{4}} z^{-1})^n - \frac{1}{2j} \sum_{n=0}^{\infty} (\frac{1}{3} e^{-j\frac{\pi}{4}} z^{-1})^n$$

First term requires $|(1/3)e^{j\pi/4}z^{-1}|<1 \rightarrow |z|>1/3$ since magnitude of $e^{j\pi/4}$ is 1.
second term requires $|(1/3)e^{-j\pi/4}z^{-1}|<1 \rightarrow |z|>1/3$ since magnitude of $e^{-j\pi/4}$ is 1.

Therefore X(z) converges when |z|>1/3 and applying infinite sum relationship we get z-transform:

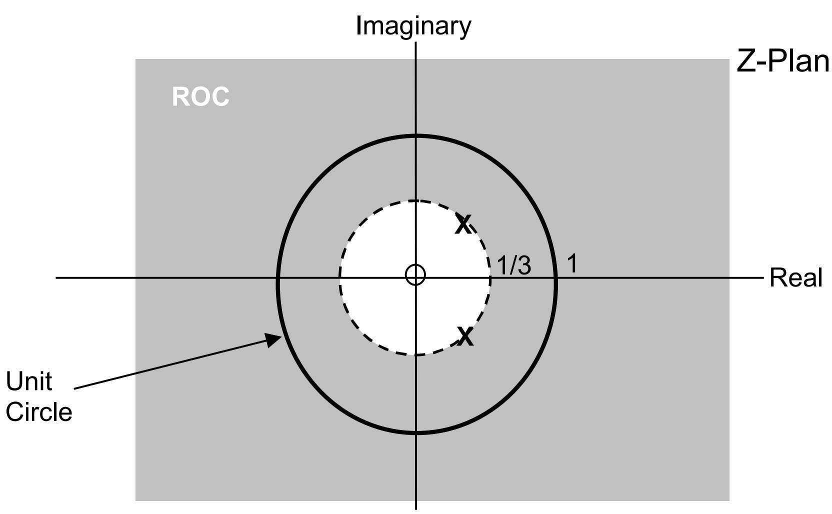
$$X(z) = \frac{1}{2j} \frac{7}{1 - \frac{1}{3} e^{j\frac{\pi}{4}} z^{-1}} - \frac{1}{2j} \frac{7}{1 - \frac{1}{3} e^{-j\frac{\pi}{4}} z^{-1}} \; where \; |z| > 1/3$$

combining the fraction we get a rational function:

$$X(z) = \frac{\frac{1}{3\sqrt{2}} Z}{(z - \frac{1}{3} e^{j\frac{\pi}{4}})(z - \frac{1}{3} e^{-J\frac{\pi}{4}})} \; where \; |z| > 1/3$$

Two Zeros: 0
Two poles: $(1/3)e^{j\pi/4}$ & $(1/3)e^{-j\pi/4}$

- Pole-zero plot and ROC are shown below:



- Some notational comment when X(z) is written as ratio of polynomials:
  - X(z) is said to have poles at infinity if degree of z in numerator exceeds the degree of the denominator.
  - X(z) is said to have zeroes at infinity if the degree of z in denominator exceeds the degree of the numerator.

9.4. Z-Transform Properties

Z-Transform has properties that are similar to the Fourier Transform. The following table contains the most common properties of Z Transform:

Property	Time-Domain $x[n]$, $x_1[n]$, $x_2[n]$	Laplace s-Domain $X(z)$, $X_1(z)$, $X_2(z)$	ROC R, R_1, R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$
Time shifting	$x[n - k]$	$z^{-k}X(z)$	R *"except for origin"*
Scaling in the z-Domain	$e^{jw_0 n}x[n]$	$X(e^{jw_0 n}z)$	R
	$z_0^n x[n]$	$X(\frac{z}{z_0})$	Z_0R
	$a^n x[n]$	$X(a^{-1}z)$	Scaled R *"|a|Z in R"*
Time reversal	X[-n]	$X(z^{-1})$	R^{-1}
Time expansion	$X_{(k)}[n] = x[p]$ where $n=pk$ 0 where $n \neq pk$ "p is an integer"	$X(z^k)$	R^{-k}
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
Conjunction	$x^*[n]$	$X^*(z^*)$	R
Difference	$X[n] – x[n - 1]$	$(1 – z^{-1})X(z)$	At least $R \cap (\|z\|>0)$
Accumulation	$\sum_{i=-\infty}^{n} x[i]$	$\frac{1}{1-z^{-1}}X(z)$	At least $R \cap (\|z\|>1)$
Differentiation in z	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint_C X_1(v)X_1(\frac{z}{v})v^{-1}dv$	
Initial Value Theorem	$x[0] = \lim_{z\to\infty} X(z)$ when x[n] is causal "x[n]=0 for n<0"		
Final Value Theorem	$x[\infty] = \lim_{z\to 1}(1-z^{-1})X(z)$ only if poles of $(1 - z^{-1})X(z)$ are inside the unit circle		

Note:

∩ "Intersection of sets" resulting set includes only the common elements

U "Union of sets" resulting set includes all the elements

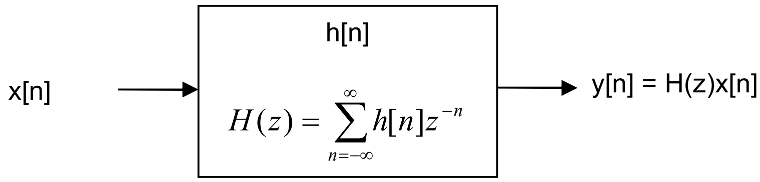
To proof any of these properties, apply the Z transform or Z inverse transform equations.

The following table contains the Z-Transform of common signals and corresponding Regions of Convergence (ROC):

Function	Time Domain, x(t)	Laplace s-domain, X(s)	ROC
Unit Impulse	$\delta[n]$	1	All z
Ideal Delay	$\delta[n - n_0]$	z^{-n_0}	$Z \neq 0$
Unit Step	u[n]	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
Unit Step	-u[-n-1]	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
Ramp	nu[n]	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
Ramp	-nu[-n-1]	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| < 1$
n^{th} Power	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
n^{th} Power	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
n^{th} Power	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
n^{th} Power	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
Cosine	$[\cos w_0 n]u[n]$	$\frac{1-[\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
Sine	$[\sin w_0 n]u[n]$	$\frac{[\sin w_0]z^{-1}}{1-[2\sin w_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
nth Power Cosine	$r^n[\cos w_0 n]u[n]$	$\frac{[r\cos w_0]z^{-1}}{1-[2r\cos w_0]z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
nth Power Sine	$r^n[\sin w_0 n]u[n]$	$\frac{[r\sin w_0]z^{-1}}{1-[2r\sin w_0]z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

9.5. Application of Z-Transform in LTI Systems

Z-transformation applies to Discrete-Time linear time-Invariant system with impulse response h[n], the system response y[n] to x[n] as shown below:



9.6. Additional Resources

- Oppenheim, A. Signals & Systems (1997) Prentice Hall
  Chapter10

9.7. Problems

Refer to www.EngrCS.com or online course page for complete solved and unsolved problem set.

Appendix A. Additional Resources

- Additional resources are available at the author’s website http://www.EngrCS.com/

- Future Enhancements
  - Add study questions, practical problems.
  - Expand Laplace and Z transform chapters.