Signals & Systems - Chapter 1

1S. Express each of the following complex numbers in Cartesian form (x + jy):

$$\frac{1}{2}e^{j\pi}, \frac{1}{2}e^{-j\pi}, e^{j\pi/2}, e^{-j\pi/2}, e^{j5\pi/2}, \sqrt{2}e^{j\pi/4}, \sqrt{2}e^{j9\pi/4}, \sqrt{2}e^{-j9\pi/4}, \sqrt{2}e^{-j\pi/4}.$$

Solution:

Theory use Euler's Rule $e^{-ja} = \cos a \pm j\sin a$

$$\frac{1}{2}e^{j\pi} = \frac{1}{2}(\cos\pi + j\sin\pi) = -\frac{1}{2}$$

$$\frac{1}{2}e^{-j\pi} = \frac{1}{2}(\cos\pi - j\sin\pi) = -\frac{1}{2}$$

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$e^{j5\pi/2} = e^{j(\pi/2+2\pi)} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$\sqrt{2}e^{j\pi/4} = \sqrt{2}\{\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\} = 1 + j$$

$$\sqrt{2}e^{j9\pi/4} = \sqrt{2}e^{j(\pi/4+2\pi)} = \sqrt{2}\{\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\} = 1 + j$$

$$\sqrt{2}e^{-j9\pi/4} = \sqrt{2}e^{-j(\pi/4+2\pi)} = \sqrt{2}\{\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\} = 1 - j$$

$$\sqrt{2}e^{-j\pi/4} = \sqrt{2}\{\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\} = 1 - j$$

1U. Express each of the following complex numbers in Cartesian form (x + jy):

$$\frac{1}{2}e^{j\pi/6}, \sqrt{3}e^{j\pi/3}, \sqrt{4}e^{j9\pi/3}, \sqrt{2}e^{-j9\pi/3}, \sqrt{2}e^{-j\pi/2}.$$

Solution:

2S. Determine the value of P_∞ and E_∞ for each of the following signals:

(a) $x_1(t) = e^{-2t}u(t)$ **(b)** $x_2(t) = e^{j(2t+\pi/4)}$ **(c)** $x_3(t) = \cos(t)$

(d) $x_1(n) = (1/2)^n u[n]$ **(e)** $x_2(n) = e^{j(\pi/2n+\pi/8)}$ **(f)** $x_2(n) = \cos(n\pi/4)$

Solution:

Theory:

Continuous → $E_\infty = \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2\, dt; \quad P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\, dt$

Discrete → $E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N}|x[n]|^2 \quad P_\infty = \lim_{T\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2$

a)

$$E_\infty = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\int_{0}^{T}e^{-4t}dt = -\frac{1}{4}(e^{-4t}|_0^\infty = \frac{1}{4}$$

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt$$

Average power can be calculated or since $E_\infty < \infty$ then P_∞ is equal to 0.

b)

$$|x(t)| = |e^{j(2t+\pi/4)}| = 1$$

$$E_\infty = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\int_{-T}^{T}1dt = \infty$$

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}1dt = \lim_{T\to\infty}\frac{T-(-T)}{2T} = 1$$

c)

$$|x(t)| = |\cos(t)| = \cos(t)$$

$$E_\infty = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\int_{-T}^{T}\cos^2(t)dt = \lim_{T\to\infty}\int_{-T}^{T}\frac{1+\cos(2t)}{2}\,dt = \infty$$

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1+\cos(2t)}{2}\,dt = \lim_{T\to\infty}\frac{1}{4T}(t+\frac{1}{2}\sin(2T)|_{-T}^{T} = \lim_{T\to\infty}\frac{1}{4T}(2T+\sin(2T)) = \frac{1}{2}$$

d)

$$|x[n]| = (\frac{1}{2})^{n}\ \mathrm{u[n]}$$

$$E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\sum_{n=0}^{N}(\frac{1}{4})^n = \frac{1}{1-1/4} = \frac{4}{3}$$

$$P_\infty = 0 \text{ since } E_\infty < \infty$$

Note: Infinite Geometric Series when |r|<1 → $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

e)

$$|x[n]| = |e^{j(\pi n/2+\pi/8)}| = 1$$

$$E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N}1 = \infty$$

$$P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|1 = \lim_{N\to\infty}\frac{2N+1}{2N+1} = 1$$

f)

$$|x[n]| = |\cos(\frac{\pi}{4}n)| = \cos(\frac{\pi}{4}n)$$

$$E_\infty = \lim_{N\to\infty}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N}\cos^2(\frac{\pi}{4}n) = \infty$$

$$P_\infty = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|\cos^2(\frac{\pi}{4}n)| = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\left(\frac{1+\cos(\frac{\pi}{2}n)}{2}\right)$$

$$P_\infty = \lim_{N\to\infty}\frac{1}{2(2N+1)}\{\sum_{n=-N}^{N}(1) + \sum_{n=-N}^{N}(\cos(\frac{\pi}{2}n))\} = \lim_{N\to\infty}\frac{(2N+1)}{2(2N+1)} = \frac{1}{2}$$

Note – The above simplifications use the following equalities:

$$\cos^2(a) = \frac{1+\cos(2a)}{2} \qquad \sum_{n=-N}^{N}\cos(\frac{\pi}{2}n) = 1$$

2U. Determine the value of P_∞ and E_∞ for each of the following signals:

(a) $x_1(t) = 5e^{-2t}\,u(t-2)$ **(b) $x_2(t) = e^{j(2t - \pi/4)}$**

(c) $x_3[n] = 3\sin(n\pi/4)$ **(d) $x_1[n] = (1/2)^n\,u[n-6]$**

Solution:

3S. Let x(t) be a signal with x(t) =0 for t < 3. For each signal given below, determine the values of t for which it is guaranteed to be zero.

(a) x(1 - t) **(b) x(1 – t) + x(2 – t)** **(c) x(1 – t)x(2 – t)**

(d) x(3t) **(e) x(t/3)**

Solution:

a) x(1 – t) → (1-t) < 3 → t > -2
Note: this entails flipping the function and then shifting by 1 to the right.

b) x(1-t) First (1-t) < 3 → t > -2
Note: this entails flipping the function and then shifting by 1 to the right.
x(2-t) First (2-t) < 3 → t > -1
Note: this entails flipping the function and then shifting by 2 to the right.

So the combined function is zero when t > -1

c) x(1-t) First (1-t) < 3 → t > -2
Note: this entails flipping the function and then shifting by 1 to the right.
x(2-t) First (2-t) < 3 → t > -1
Note: this entails flipping the function and then shifting by 2 to the right.

So the combined function is zero when t > -2

d) x(3t) First (3t) < 3 → t <1
Note: this entails compressing the function by a factor of 3 linearly

So the compressed function is zero when t < 1

e) x(t/3) First (t/3) < 3 → t <9
Note: this entails expressing the function by a factor of 3 linearly

So the expanded function is zero when t < 9

3U. Let x(t) be a signal with x(t) =0 for t > 1. For each signal given below, determine the values of t for which it is guaranteed to be zero (if any).

(a) x(1 - t) **(b) x(1 – t) + x(2 – t)** **(c) x(1 – t)x(2 – t)**

(d) x(3t) **(e) x(t/3)**

Solution:

4S. Determine the fundamental period of the signal x(t) = 2cos(10t + 1) – sin(4t – 1).

Solution:

First term: $2\pi f_0 = 10$ → $f_0 = 10/2\pi = 5/\pi$ → Fundamental Period = $T_0 = \pi/5$
Second term: $2\pi f_0 = 4$ → $f_0 = 4/2\pi = 2/\pi$ → Fundamental Period = $T_0 = \pi/2$

For the overall signal periodic signal's fundamental period must have the least common multiple of the first and second term → $10\pi/10 = \pi$

4U. Determine the fundamental period of the signal x(t) = 2sin(12t + 6) – cos(3t – 3).

Solution:

5S. Determine the fundamental period of the signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$

Solution:

First term is DC so the period is 0

Second term: $N_0 = (2\pi/w_o)m$, with $w_o = 4\pi/7$ → $N_0 = m(7/2)$ → Fundamental Period = $N_0 = 7$ (where m=2)

Note: m is selected such that N_o is the smallest possible integer.

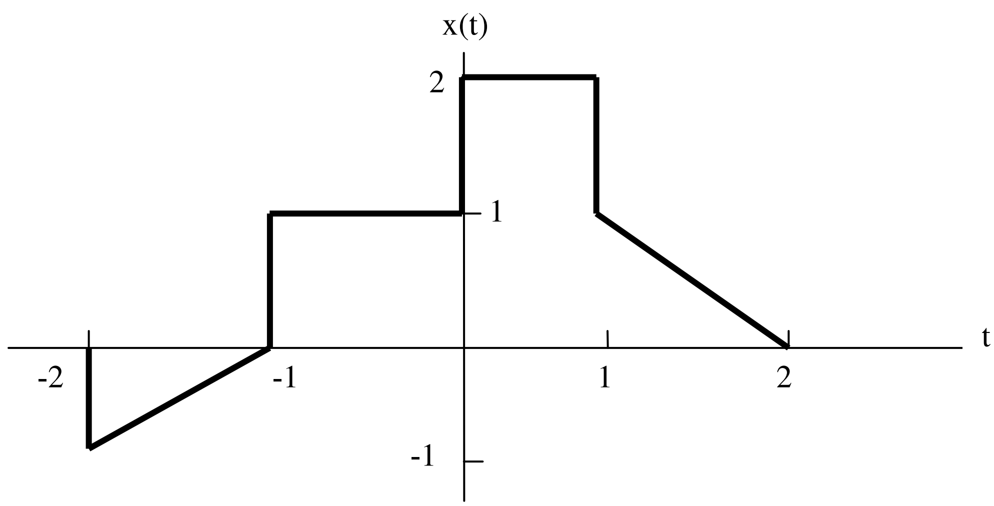
Third term: $N_0 = (2\pi/w_o)m$, with $w_o = 2\pi/5$ → $N_0 = m(5)$ → Fundamental Period = $N_0 = 5$ (where m=1)

For the overall signal x[n] is periodic with a period which is the least common multiple of the three terms which is equal to 35.

5U. Determine the fundamental period of the signal $x[n] = 1 + e^{j6\pi n/5} - e^{j8\pi n/7}$

Solution:

6S. A continuous-time signal x(t) is shown in the following figure.



Sketch and label carefully each of the following signals:

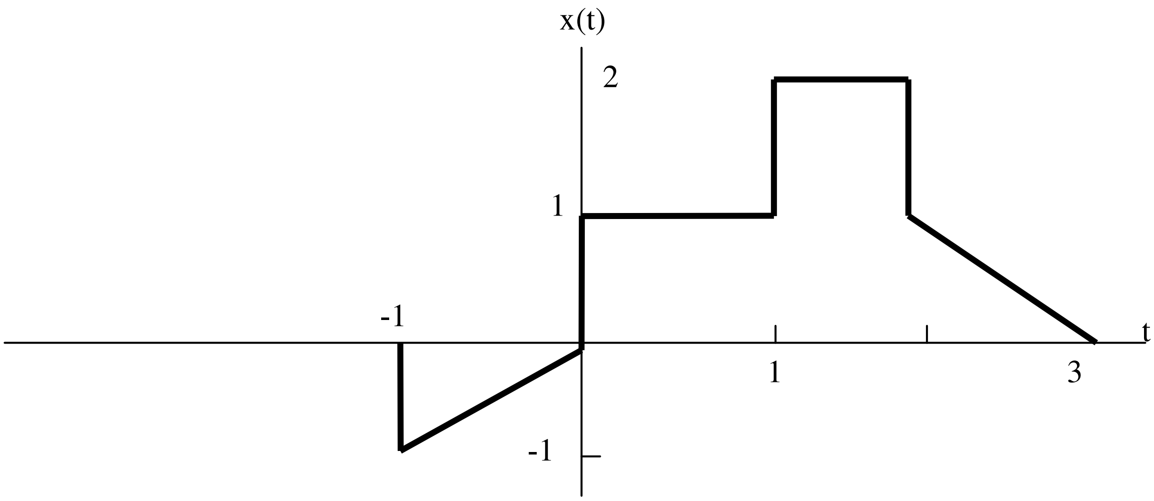
(a) x(t – 1) **(b) x(2 – t)** **(c) x(2t + 1)**

(d) x(4 – t/2) **(e) [x(t) + x(-t)]u(t)** **(f) x(t)[δ(t + 3/2) - δ(t - 3/2)]**

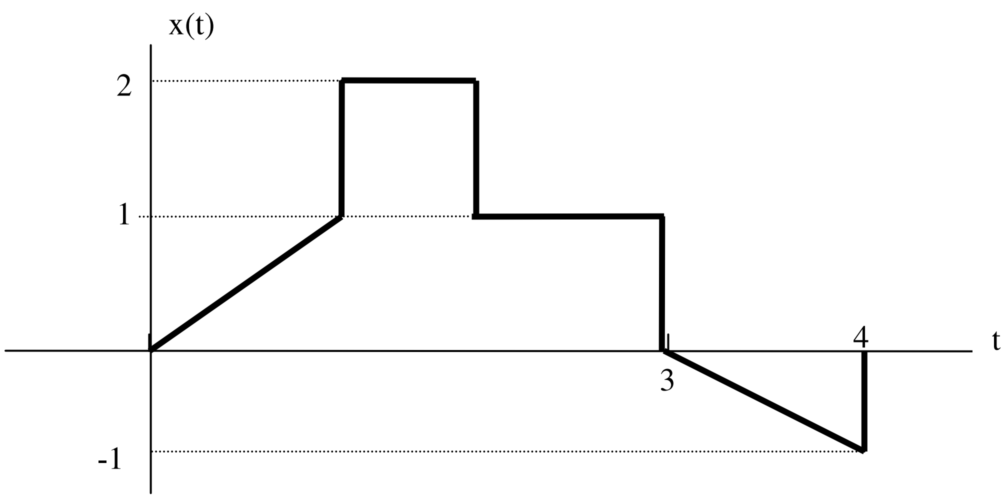
Solution:

Note: Order of operation is Shift, flip, Expand/Compress

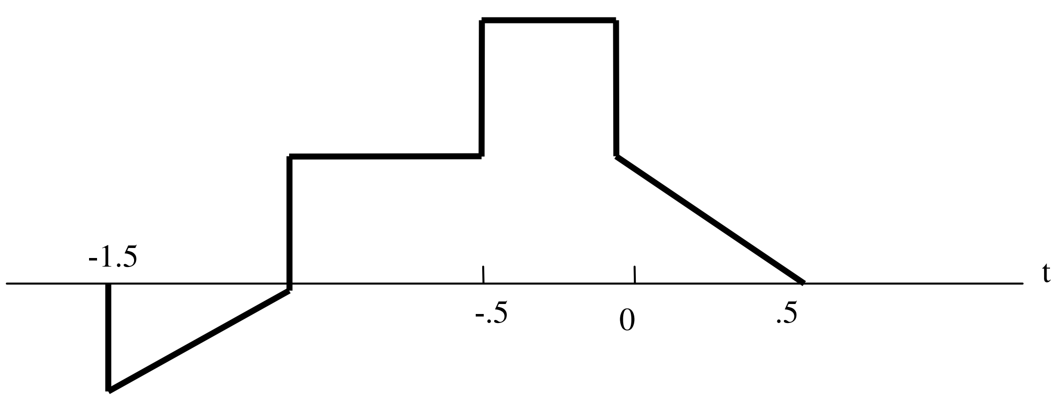
a) x(t-1) shift x(t) to right by 1.



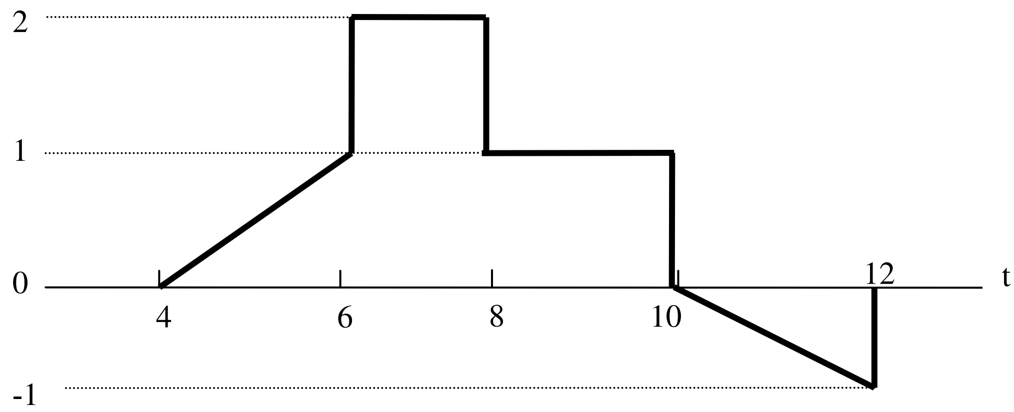
b) x(2-t), shift x(t) left by 2 and flip



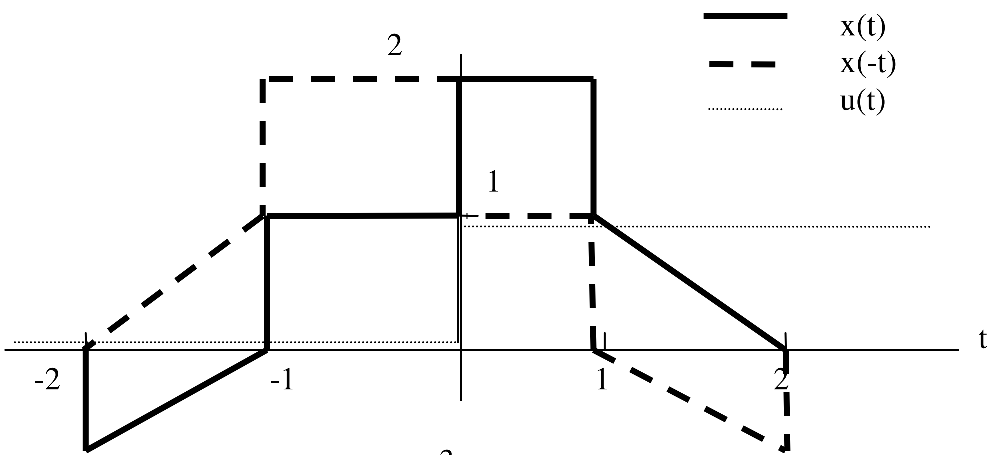
c) x(2t+1), shift left by 1 and compress by 2 .

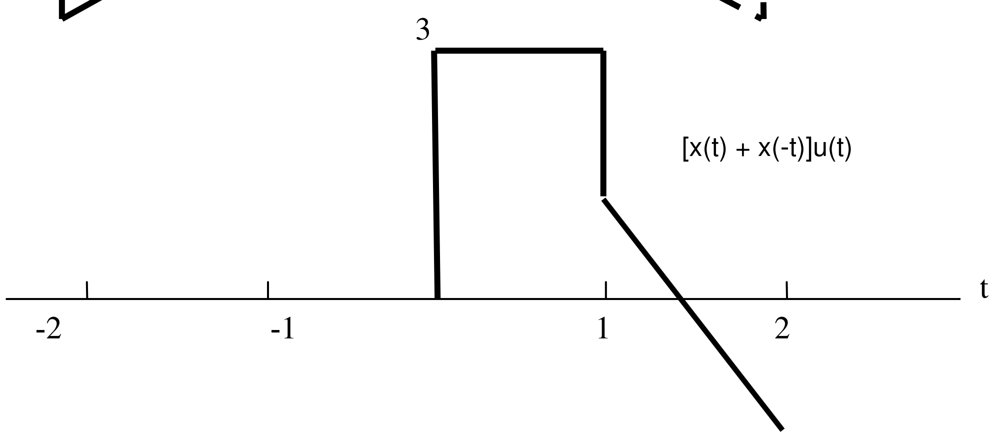


d) x(4 – t/2), shift to right by 4, Flip and expand by 2.

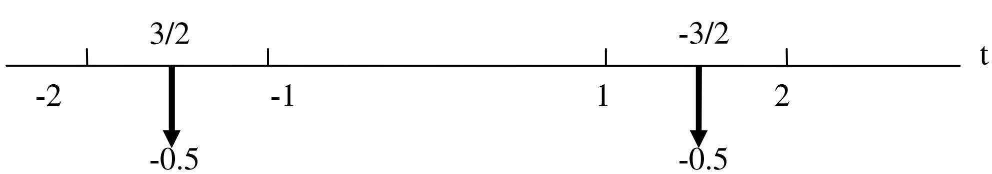


e) [x(t) +x(-t)]u(t)

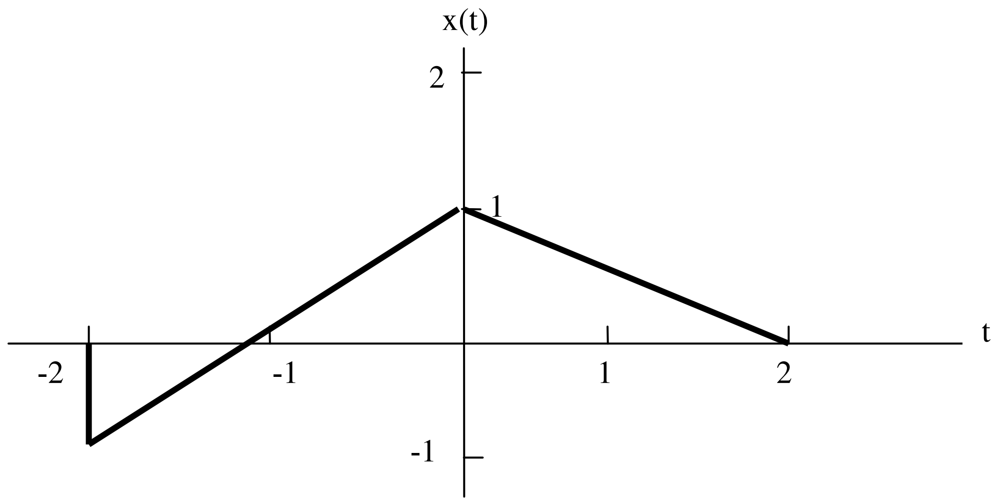




f) x(t)[δ(t + 3/2) - δ(t - 3/2)]
at t=-3/2 → x(-3/2)
at t=3/2 → x(3/2)



6U. A continuous-time signal x(t) is shown in the following figure.



Sketch and label carefully each of the following signals:

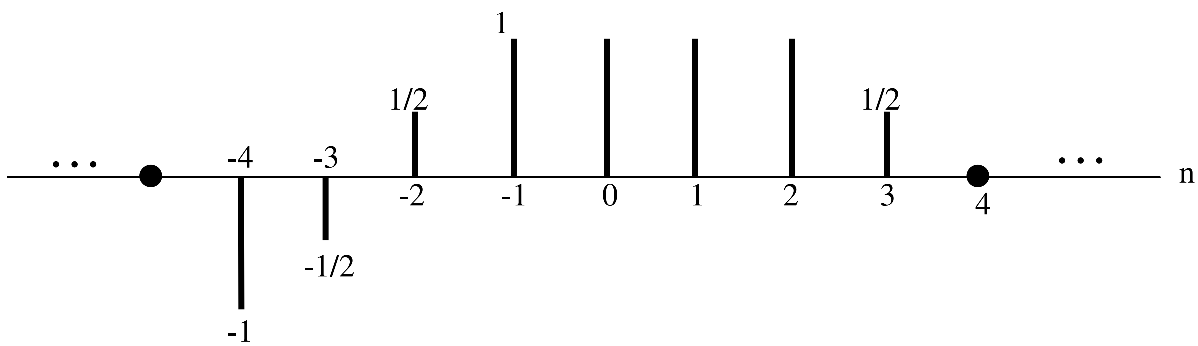
(a) x(t – 2) **(b) x(4 – t)**

(c) [x(2t) + x(-2t)]u(t) **(d) x(t)[δ(t + 1/2) - δ(t - 1/2)]**

Solution:

Note: Order of operation is Shift, flip, Expand/Compress

7S. A discrete-time signal shown below.



Sketch and label carefully each of the following signals:

(a) x[n – 4] **(b) x[3 – n]** **(c) x[3n]**

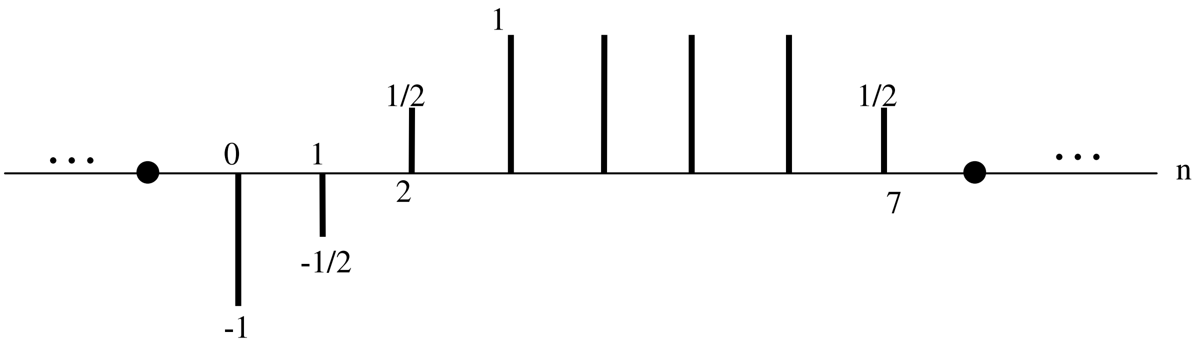
(d) x[3n + 1] **(e) x[n]u[3 – n]** **(f) x[n – 2] δ[n – 2]**

(g) ½ x[n] + ½ $(-1)^n$x[n] **(h) x[$(n – 1)^2$]**

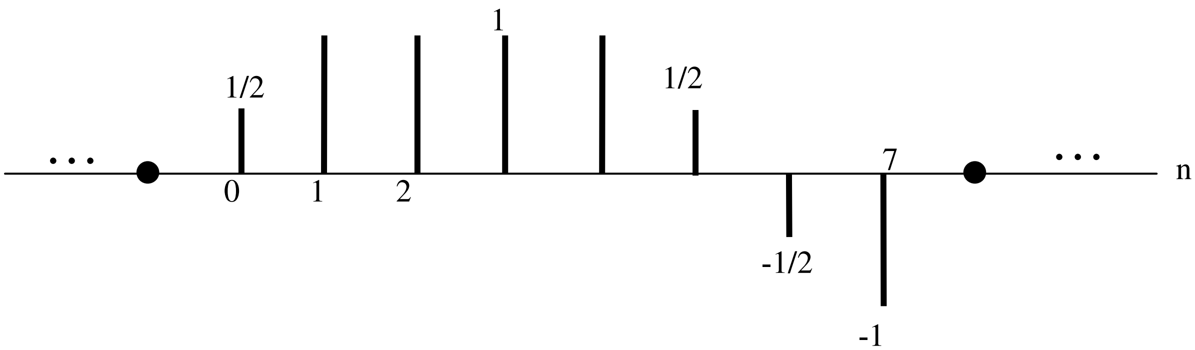
Solution:

Note: Order of operation is Shift, flip, Expand/Compress

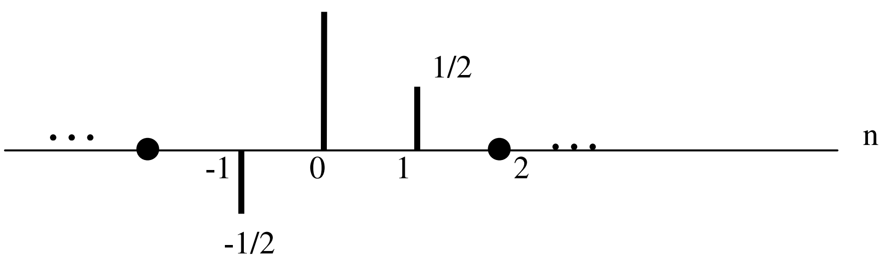
a) x[n -4] shift the signal to the right by 4



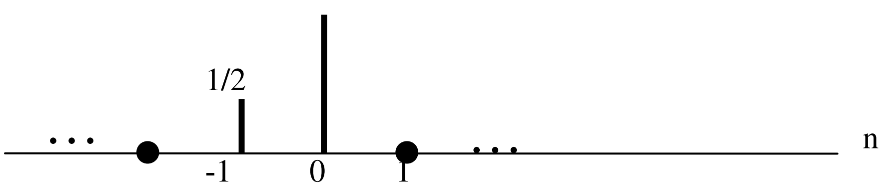
b) x[3 - n] Flip signal and shift the signal to the right by 3



c) x[3n] Compress the signal by factor of three (all the non integer are not seen)

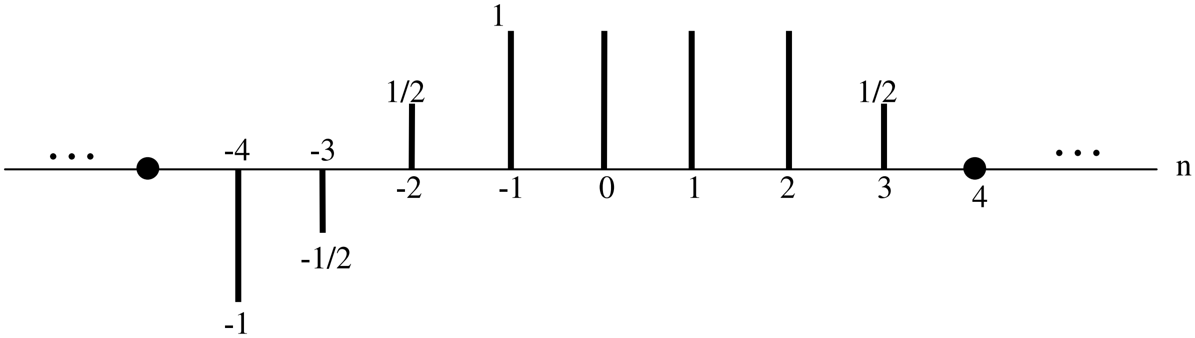


d) x[3n + 1] shift the signal to left by 1 and then Compress by 3 (all the non integer are not seen) →



e) x[n]u[3 - n] → x[n]
u[3-n] = 1 for 3-n ≥ 0 → n ≤ 3

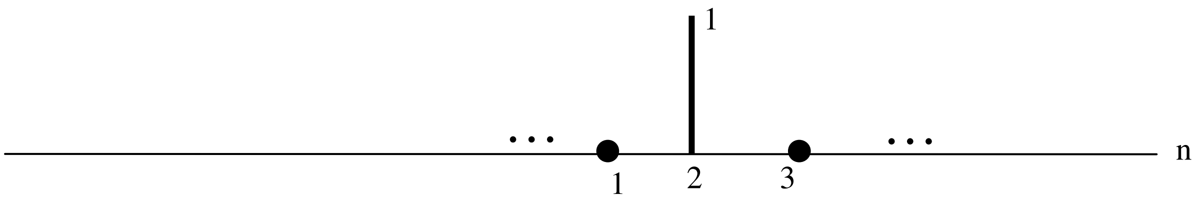
0 Otherwise



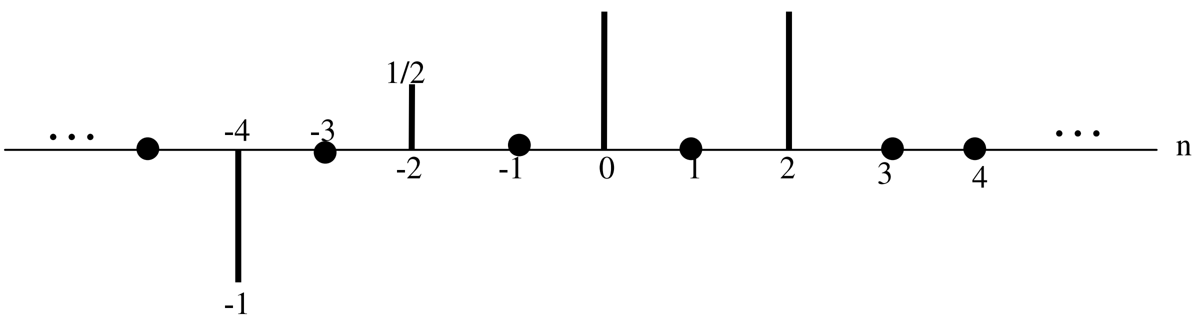
f) $x[n - 2]\delta[n-2] \rightarrow$

= X[0] for n=2

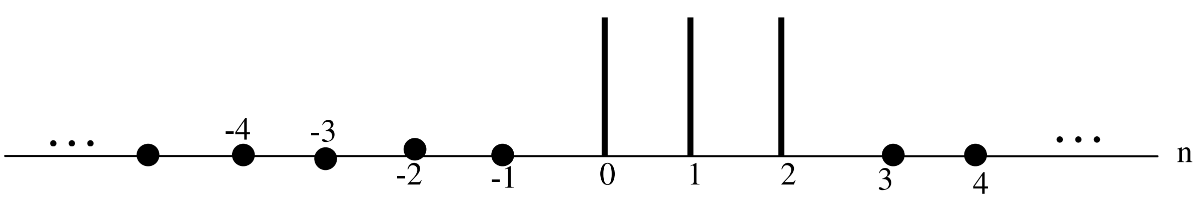
0 Otherwise



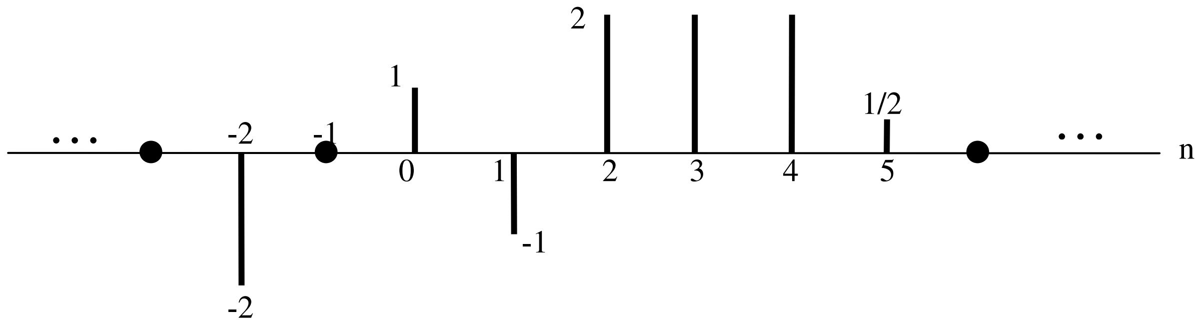
g) $(1/2)x[n] + (1/2)(-1)^n x[n]$



h) $x[(n-1)^2]$



7U. A discrete-time signal shown below.



Sketch and label carefully each of the following signals:

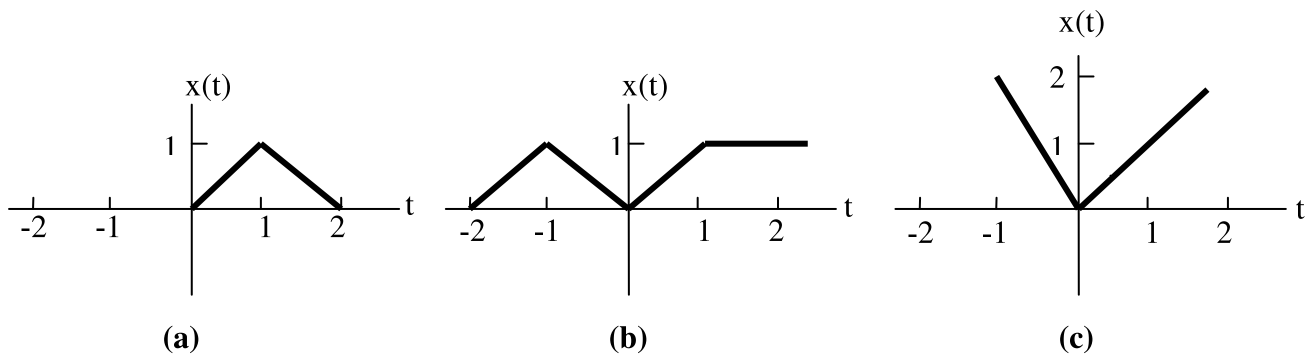
(a) $x[n + 3]$ **(b) $x[6 - n]$** **(c) $x[2n]$**

(d) $x[n]u[4 - n]$ **(e) $x[n - 2]\,\delta[n - 3]$** **(f) $x[(n - 2)^2]$**

Solution:

Note: Order of operation is Shift, flip, Expand/Compress

8S. Determine and sketch the even and odd part of the signals depicted below. Label your sketches carefully.



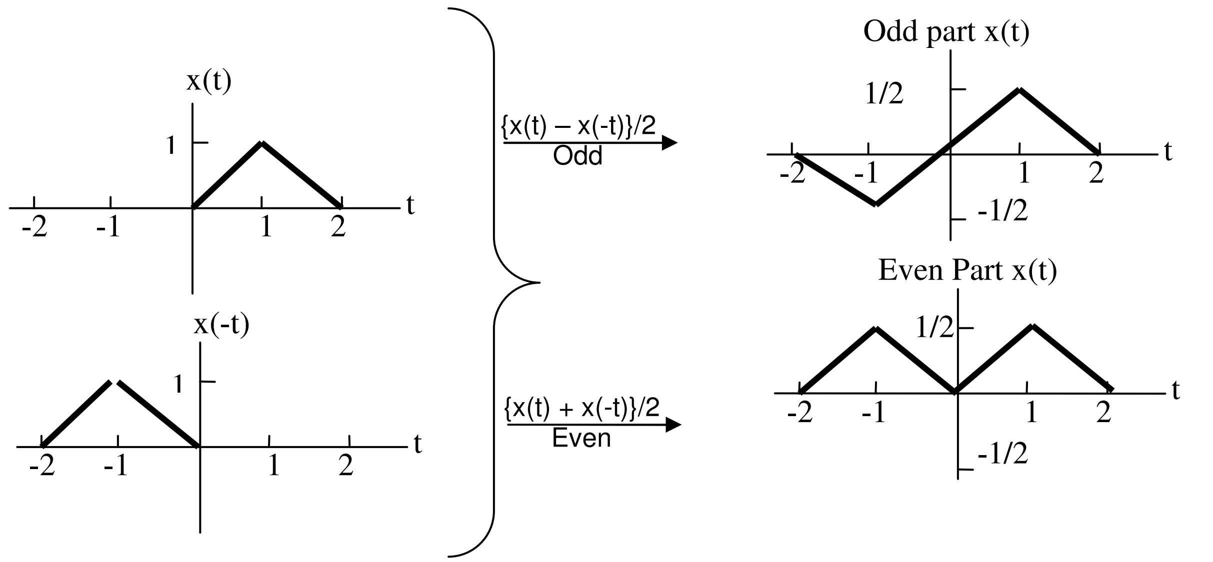
Solution:

Theory →

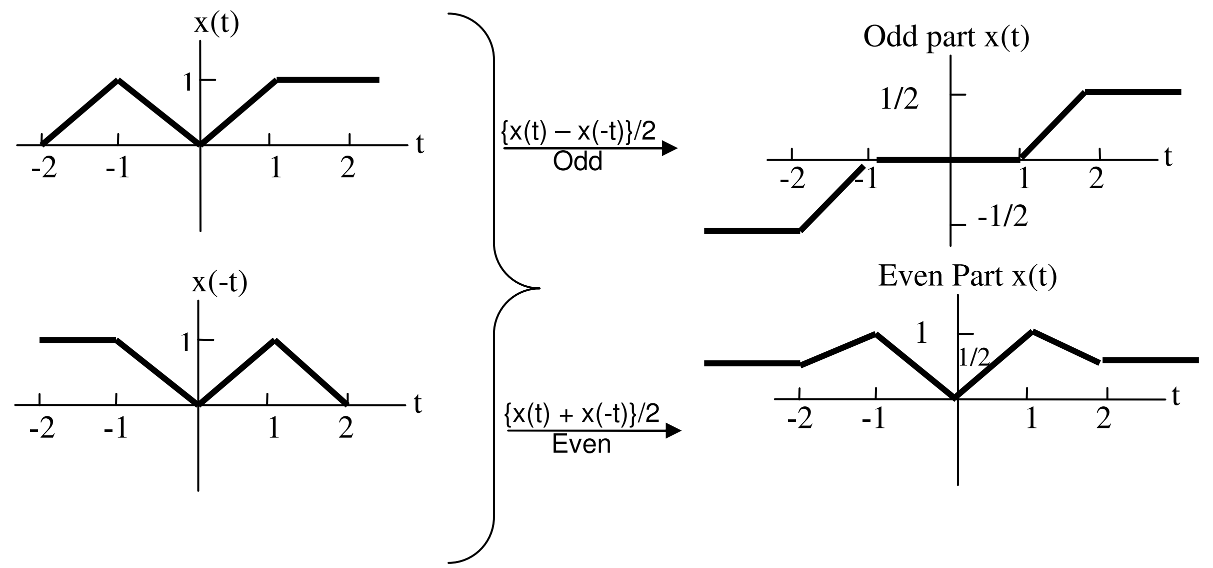
Odd part of x(t) = $\{x(t) - x(-t)\} / 2$

Even part of x(t) = $\{x(t) + x(-t)\} / 2$

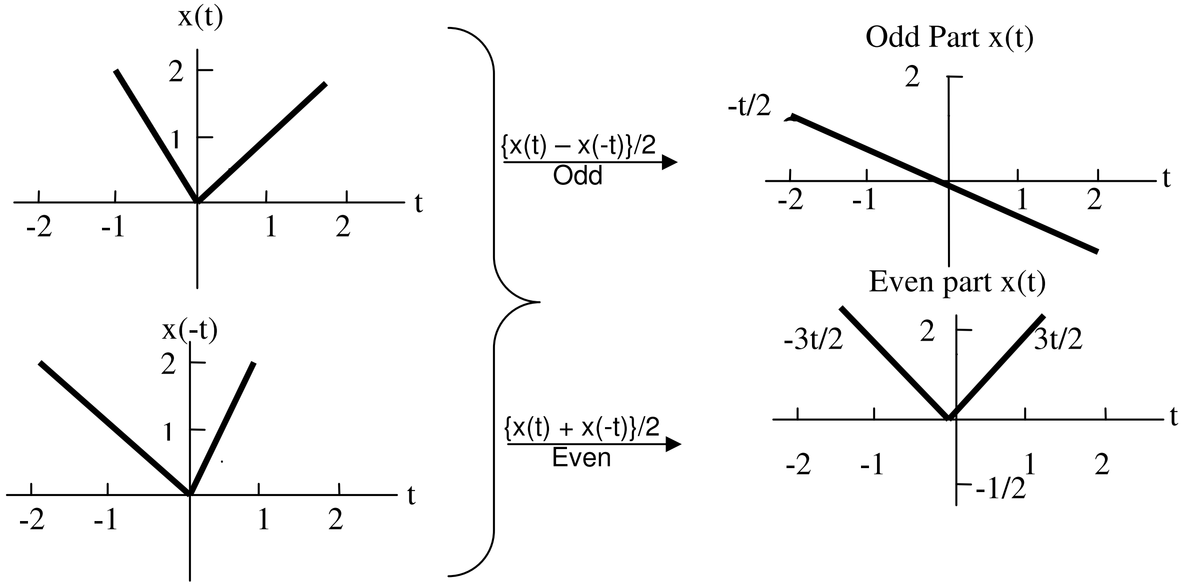
a)


x(t)
x(-t)
{x(t) – x(-t)}/2
Odd
{x(t) + x(-t)}/2
Even
Odd part x(t)
Even Part x(t)
1/2
-1/2
1
-2
-1
1
2
t

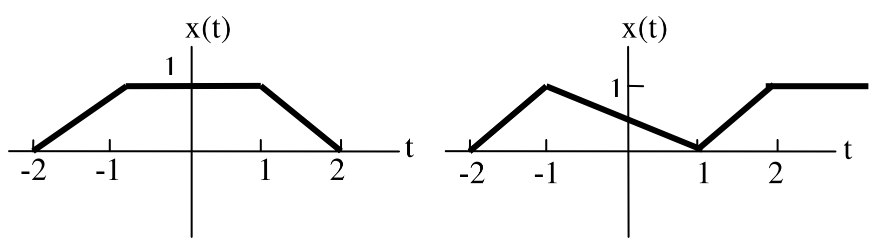
b)


x(t)
x(-t)
{x(t) – x(-t)}/2
Odd
{x(t) + x(-t)}/2
Even
Odd part x(t)
Even Part x(t)
1/2
-1/2
1
-2
-1
1
2
t

c)



8U. Determine and sketch the even and odd part of the signals depicted below. Label your sketches carefully.



(a) **(b)**

Solution:

9S. The following general system properties were introduced:

(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable

Determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, y(t) denotes the system output and x(t) is the system input.

a) $y(t) = x(t-2) + x(2-t)$

b) $y(t) = [\cos(3t)]x(t)$

c) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$

d) $y(t) = \begin{cases} 0 & t < 0 \\ x(t) + x(t-2) & t \geq 0 \end{cases}$

e) $y(t) = \begin{cases} 0 & x(t) < 0 \\ x(t) + x(t-2) & x(t) \geq 0 \end{cases}$

f) $y(t) = x(t/3)$

g) y(t) = $\frac{dx(t)}{dt}$

Solution:

Theory:

A Memory-less System, Output only depends on input at the current time
A system is time invariant if the characteristic of the system is fixed overtime. $y(x(t-t_0)) = y(t-t_0)$
A linear system is system that possesses the property of superposition $y_1(t) + y_2(t) = y(x_1(t)+x_2(t))$
A system is causal if the output at anytime depends only on value of the input present at the time and in the past
Stable system Bounded input leads to bounded output.

	Memory less	Time Invariant	Linear	Causal	Stable
a) y(t) = x(t – 2) + x(2 – t)	NO	YES	YES	NO	YES
b) y(t) = [cos(3t)]x(t)	YES	NO	YES	YES	YES
c) y(t) = $\int_{-\infty}^{2t} x(\tau)d\tau$	NO	NO	YES	NO	NO
d) y(t) = 0 t < 0 x(t) + x(t – 2) t ≥ 0	NO	YES	YES	YES	YES
e) y(t) = 0 x(t) < 0 x(t) + x(t – 2) x(t) ≥ 0	NO	Yes	YES	YES	YES
f) y(t) = x(t/3)	NO	YES	YES	NOT	YES
g) y(t) = $\frac{dx(t)}{dt}$	NO	YES	YES	YES	NO

9U. The following general system properties were introduced:

(1) Memoryless
(2) Time invariant
(3) Linear
(4) Causal
(5) Stable

determine which of these properties hold and which do not hold for each of the following continuous-time systems. Justify your answers. In each example, y(t) denotes the system output and x(t) is the system input.

a) y(t) = x(t + 4) + x(3 – t)
b) y(t) = [sin(5t)]x(t)
c) y(t) = $\int_{3t}^{+\infty} x(\tau)d\tau$
f) y(t) = 0 t < 0
x(-t) + x(t + 2) t ≥ 0

Solution: