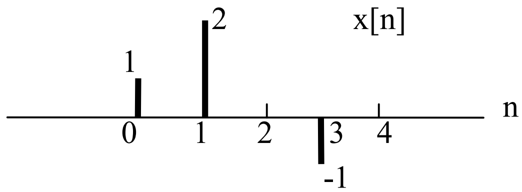
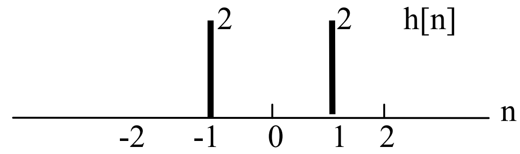
Signals & Systems - Chapter 2

1S. Let x[n] = δ[n] + 2δ[n – 1] – δ[n – 3] and h[n] = 2δ[n + 1] + 2δ[n – 1]
Compute and plot each of the following convolutions:
a) y_1[n] = x[n] * h[n] **b) y_2[n] = x[n+2] * h[n]** **c) y_3[n] = x[n] * h[n+2]**

Solution:

a)
Theory—We have $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$





y_1[n] = h[-1]x[n+1] + h[1]x[n-1] = 2x[n+1] + 2x[n-1]
y_1[n]= 2δ[n+1] + 4δ[n] – 2δ[n – 2] + 2δ[n-1] + 4δ[n – 2] – 2δ[n – 4]
y_1[n]= – 2δ[n – 4] + 2δ[n – 2] + 2δ[n-1] + 4δ[n] + 2δ[n+1]

b) Note x[n] → x[n+2] which is a shift of +2
Since we have linear time invariant system

$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[(n+2)-k]$ → $y_2[n] = y_1[n+2]$ Therefore

y_2[n]= – 2δ[n – 2] + 2δ[n] + 2δ[n+1] + 4δ[n+2] + 2δ[n+3]

c) Note h[n] → h[n+2] which is a shift of +2
Since we have linear time invariant system
$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} h[k+2]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$ → $y_3[n] = y_1[n+2]$ Therefore
y_3[n]= – 2δ[n – 2] + 2δ[n] + 2δ[n+1] + 4δ[n+2] + 2δ[n+3]

1U. Let x[n] = δ[-n] + 2δ[n + 1] – δ[n – 4] and h[n] = δ[n + 2] + 3δ[n + 1]
Compute and plot each of the following convolutions:
a) y_1[n] = x[n] * h[n] **b) y_2[n] = x[n-2] * h[n]** **c) y_3[n] = x[n] * h[n-2]**
Solution:

2S. Consider an input x[n] and a unit impulse response h[n] given by

$$x[n] = (\frac{1}{2})^{n-2} u[n-2]$$

$$h[n] = u[n+2]$$

Determine and plot the output y[n] = x[n]*h[n].

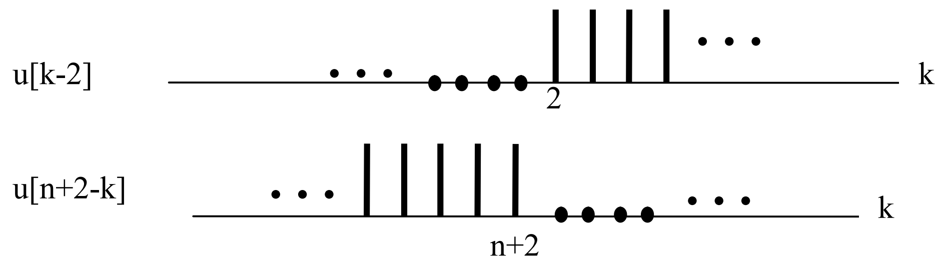
Solution:

a)

Theory—We have $y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[k-2]u[n+2-k]$$

taking a look at plot of the two step function we see that
u[k-2] = 1 when k-2≥0 or k≥2
0 otherwise
u[n+2-k] = 1 when n+2-k≥0 or k≤n+2
0 otherwise



There are two conditions:
1) when n+2 < 2 ➔ y[n] = 0 for n<0

2) when n+2 ≥ 2 ➔ $y[n] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k}$ for n≥0

Apply finite sum formula → $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ for n≥0 & 0< |a| <1

$y[n] = \frac{1-(1/2)^{n+1}}{1-1/2} u[n]$ = $2[1 – (1/2)^{n+1}]u[n]$

2U. Consider an input x[n] and a unit impulse response h[n] given by

$$x[n] = (\frac{1}{4})^{n-3} u[n+2]$$
$$h[n] = 2u[n-2]$$

Determine and plot the output y[n] = x[n]*h[n].

Solution:

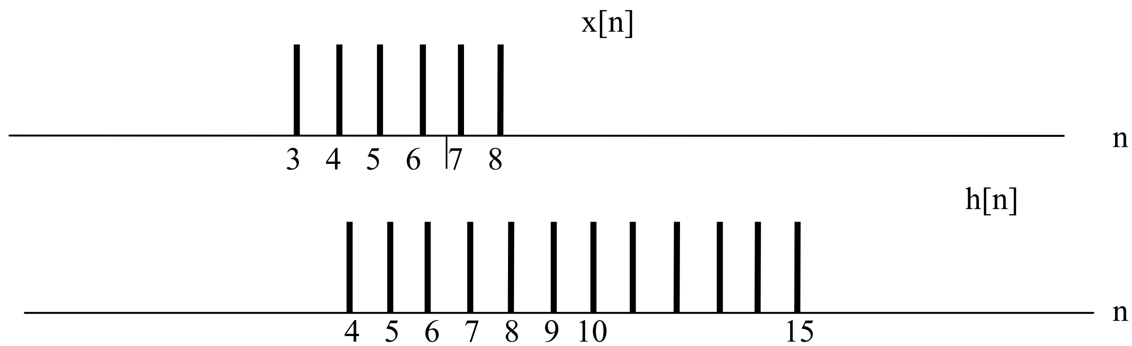
3S. Compute and plot y[n] = x[n] * h[n], where

$$x[n] = 1 \quad for \quad 3 \le n \le 8$$
$$0 \quad otherwise$$

$$h[n] = 1 \quad for \quad 4 \le n \le 15$$
$$0 \quad otherwise$$

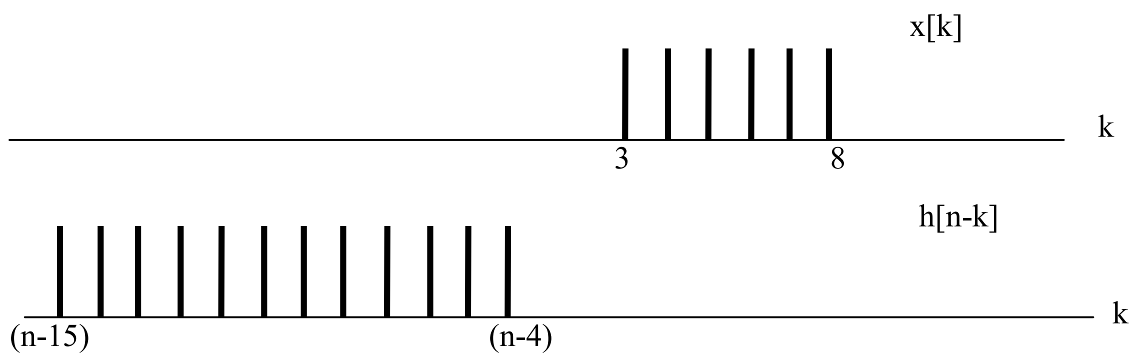
Solution:

a)



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Let's take the graphic path to find out the intervals of interest --- first need to plot h[n-k]



Look for segment where the signal have the same amount of overlap

for $3 \le n-4 \le 8$ or $7 \le n \le 12$	$y[n] = \sum_{k=3}^{n-4} 1 = n-6$	Partial front-end overlap
for $8 \le n-4$ & $n-15 \le 3$ or $12 \le n \le 18$	$y[n] = 6$	Full overlap
for $3 \le n-15 \le 8$ or $18 \le n \le 23$	$y[n] = \sum_{k=n-1}^{8} 1 = 24-n$	Partial back-end overlap
otherwise	$y[n] = 0$	or no overlap.

3U. Compute and plot y[n] = x[n] * h[n], where

$$x[n] = 1 \quad for \quad 2 \le n \le 5$$
$$0 \quad otherwise$$

$$h[n] = 1 \quad for \quad -3 \le n \le 8$$
$$0 \quad otherwise$$

Solution:

4S. Compute and plot the convolution y[n] = x[n] * h[n], where

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] \quad and \quad h[n] = u[n-1]$$

Solution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{-k} u[-k-1]u[n-k-1]$$

u[-k-1] = 1 for –k-1 ≥0 → k≤-1
0 otherwise

Condition 1 – when k≤-1 the summation is not 0 & the equation for y[n] can be writer as:

$$y[n] = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} u[n-k-1]$$

u[n-k-1] = 1 for n–k-1 ≥0 → k ≤ n-1
0 otherwise

Condition 2 – when k≤-n-1 the summation is not 0 & the equation for y[n] can be writer as:

$$y[n] = \sum_{k=-\infty}^{n-1} \left(\frac{1}{3}\right)^{-k}$$

Considering Condition 1 and 2, we have two intervals for output:

For n≥0 the above equation reduces → $y[n] = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k}$

For n<0 the above equation reduces → $y[n] = \sum_{k=-\infty}^{n-1} \left(\frac{1}{3}\right)^{-k}$

These equations can be further simplified by applying the Infinite Geometric Sum Series.

4U. Compute and plot the convolution y[n] = x[n] * h[n], where

$$x[n] = \left(\frac{1}{8}\right)^{-n} u[n+1] \quad and \quad h[n] = u[n-5]$$

Solution:

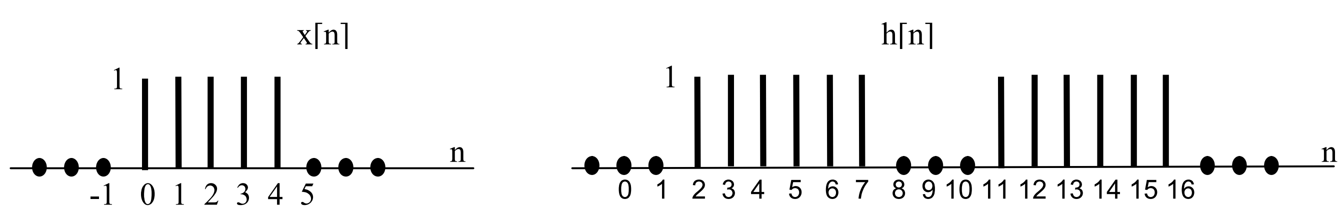
5S. Compute the convolution y[n] = x[n] * h[n] of the following pairs of signals:

a) $x[n] = \alpha^n u[n]$, $h[n] = \beta^n u[n]$, $when \alpha \neq \beta$

b) $x[n] = h[n] = \alpha^n u[n]$

c) $x[n] = (-\frac{1}{2})^n u[n-4]$, $h[n] = 4^n u[2-n]$

d) x[n] and h[n] are as in following figure



Hint: first draw x(t) and h(t). reflect h(t) about x=0 and then walk the signal to find the limits.

Solution:

a)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = \sum_{k=0}^{n} a^k \beta^{n-k} \quad for\ \ n \geq 0$$

$$y[n] = \beta^n \sum_{k=0}^{\infty} (a/\beta)^k \quad for\ \ n \geq 0$$

$$or\ apply\ \ inifinit\ \ series \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

$$y[n] = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right]u[n] \quad for\ \ \alpha \neq \beta$$

b)

Note, this part is the same as “part a” except for the fact that a=β

$$y[n] = \alpha^n \left[\sum_{k=0}^{n} (1)^k\right] u[n] = (n+1)a^n u[n]$$

c)

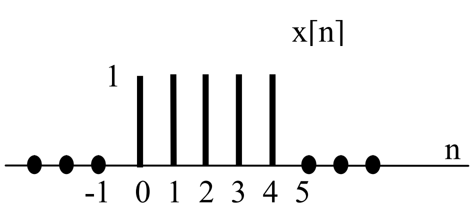
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

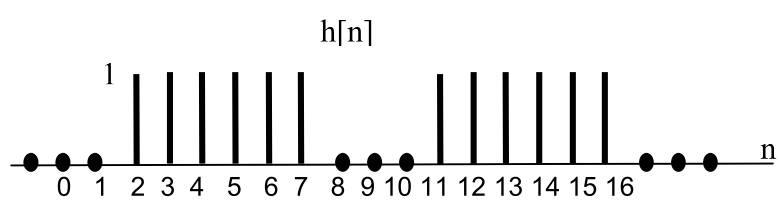
$$y[n] = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k (4)^{n-k} = (4)^n \sum_{k=4}^{\infty} \left(-\frac{1}{8}\right)^k \quad for\ \text{n-2} < 4 \rightarrow n < 6$$

$$y[n] = \sum_{k=n}^{\infty} \left(-\frac{1}{2}\right)^k (4)^{n-k} = (4)^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{8}\right)^k \quad for\ n-2 \geq 4 \rightarrow n \geq 6$$

To Further Simplify $apply\ inifinit\ series \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ $\quad \text{since } a = -1/8 \Rightarrow \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

d)

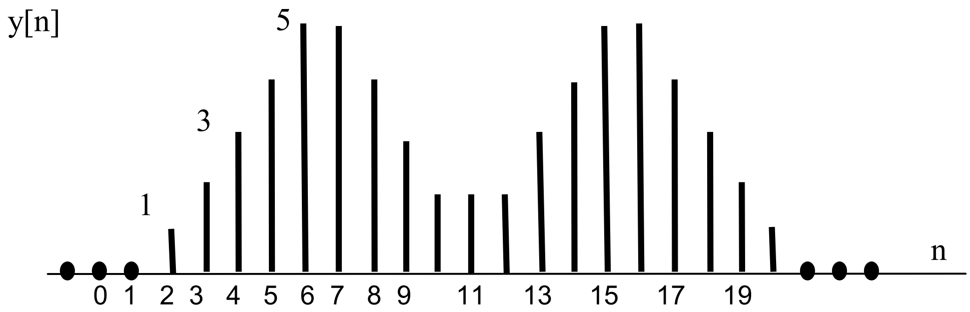




$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = x[0]h[n-0] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] + x[4]h[n-4]$$

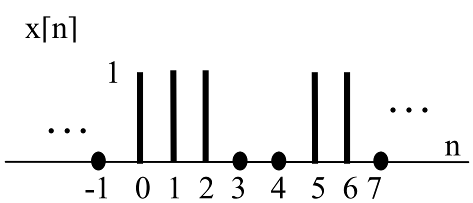
$$y[n] = h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$$

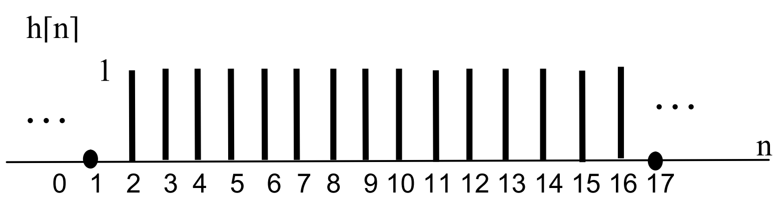


5U. **Compute the convolution y[n] = x[n] * h[n] of the following pairs of signals:**

a)
$$x[n] = (\frac{1}{3})^n u[n+2]$$
$$h[n] = 8^n u[3+n]$$

b)



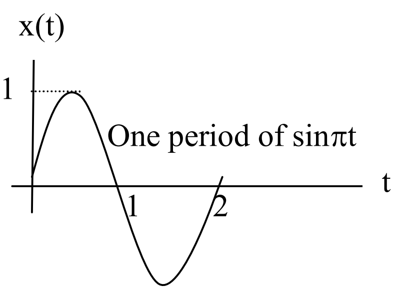


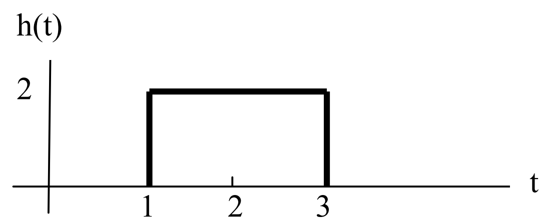
6S. For each of the following pairs of waveforms, use the convolution integral to find response y(t) of the LTI system with impulse response h(t) and x(t). Sketch your results.

a) $x(t) = e^{-\alpha t}u(t)$, $h(t) = e^{-\beta t}u(t)$ **(Do this both when** $\alpha \neq \beta$ *and* $\alpha = \beta$ **.)**

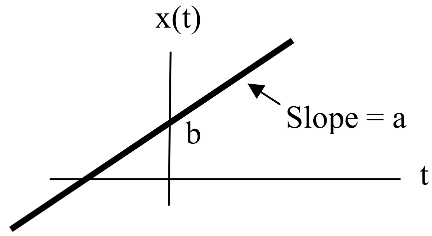
b) x(t) = u(t) – 2u(t – 2) + u(t – 5) and h(t) = e^{2t}**u(1 – t)**

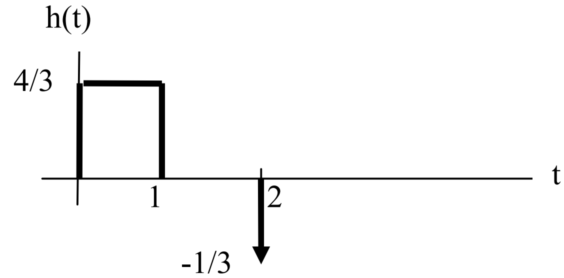
c) x(t) and h(t) shown below:



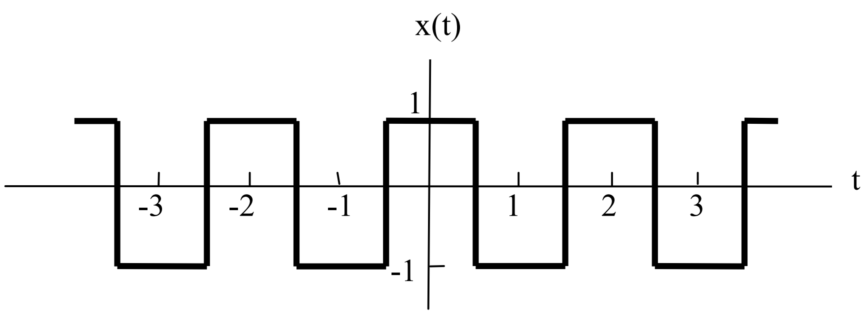


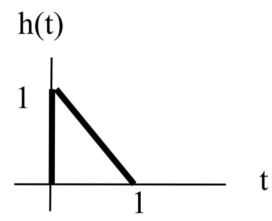
d) x(t) and h(t) shown below:





e) x(t) and h(t) shown below:





Hint: first draw x(t) and h(t). reflect h(t) about x=0 and then walk the signal to find the limits.

Solutions:

a) The desired Convolution is:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = 0(no\ overlap)\ for\ t \geq 0$$

$$y(t) = \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)} d\tau \quad for\ t \geq 0$$

$$Therefore$$

$$y(t) = e^{-\beta t}\left[\int_0^t e^{(\beta-\alpha)\tau} d\tau\right]u(t)$$

$$for \quad \alpha = \beta \Rightarrow y(t) = e^{-\beta t}\left[\int_0^t d\tau\right]u(t0 = te^{-\beta t}u(t)$$

$$for \quad \alpha \neq \beta \Rightarrow y(t) = \frac{e^{-\beta t}\left[e^{(\beta-\alpha)t} - 1\right]}{\beta - \alpha}$$

b) The desired Convolution is:
First draw x(t) and h(-t) to find the number of unique overlap sections.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Considering the intervals:

$t \ll 1 \rightarrow y(t) = \int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau$

$1 < t \leq 3 \rightarrow y(t) = \int_0^{t-1} e^{2(t-\tau)} d\tau - \int_2^5 e^{2(t-\tau)} d\tau$

$3 < t \leq 6 \rightarrow y(t) = -\int_{t-1}^{5} e^{2(t-\tau)} d\tau$

$6 \leq t \rightarrow y(t) = 0$

c) The desired Convolution is:
First draw x(t) and h(-t) to find the number of unique overlap sections.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^2 \sin(\pi t)h(t-\tau)d\tau$$

$$\textit{Considering the overlap types}:$$

$$y(t) = 0 \quad \textit{for } t \leq 1$$

$$y(t) = (2/\pi)[1-\cos\{\pi(t-1)\}] \quad \textit{for } 1 < t < 3$$

$$y(t) = (2/\pi)[1-\cos\{\pi(t-3)\}-1] \quad \textit{for } 3 < t < 5$$

$$y(t) = 0 \quad \textit{for } 5 < t$$

d) The desired Convolution is:

let $h(t) = h_1(t) - (1/3)\delta(t-2)$ where $h_1(t)$= 4/3 for 0≤t≤1
0 otherwise

so we can use the system linearity to write:
$y(t) = h(t)*x(t) = h_1(t)*x(t) - (1/3)x(t-2)$

we have

$$h_1(t) * x(t) = \int_{t-1}^{t} \frac{4}{3}(a\tau + b)\, d\tau = \frac{4}{3}\Big[(\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\Big]$$

$$Therefore$$

$$y(t) = \frac{4}{3}\Big[(\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)\Big] - \frac{1}{3}[(a(t-2) + b]$$

e) The desired Convolution is:
for LTI system when the input is periodic, it implies that output y(t) is also periodic so determine one period.
But remember to include effect of other periods on calculation of single period output.

So we will take the period -1/2 < t < 3/2
x(t)=1 for -1/2 < t < 1/2
-1 for 1/2 < t < 3/2

h(t) = t+1 for 0 < t < 1
= 0 otherwise

First draw x(τ)and h($t-\tau$) to find the number of unique overlap sections.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_{t-1}^{-1/2}(t-\tau-1)d\tau - \int_{-1/2}^{t}(t-\tau-1)d\tau = -t^2 + t + 1/4 \quad for\ -1/2 < t < 1/2$$

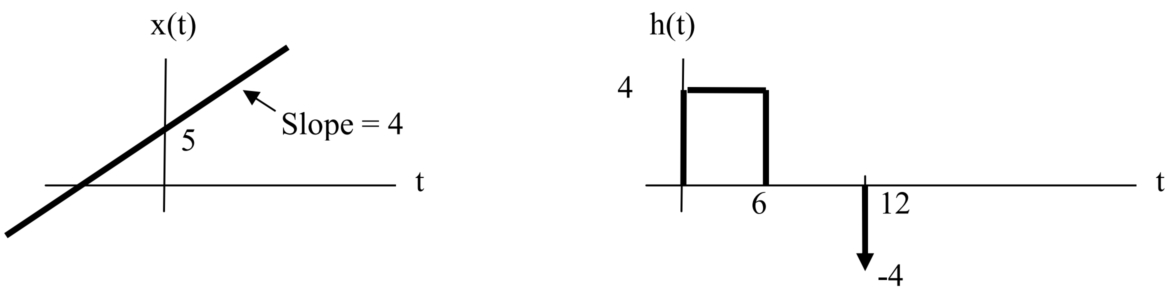
$$y(t) = -\int_{t-1}^{1/2}(t-\tau-1)d\tau + \int_{1/2}^{t}(t-\tau-1)d\tau = t^2 - 3t + 7/4 \quad for\ 1/2 < t < 3/2$$

Note: O*utput period is also 2.*

6U. For each of the following pairs of waveforms, use the convolution integral to find response y(t) of the LTI system with impulse response h(t) and x(t). Sketch your results.

a) x(t) = 3u(-t) – 5u(t + 2) + 3u(-t + 3) and h(t) = e^{-2t}u(-2 + t)

b) x(t) and h(t) shown below:



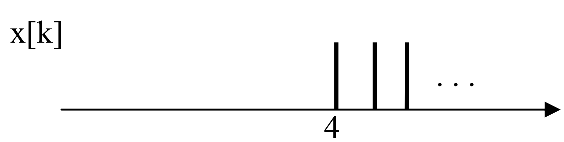
Solutions:

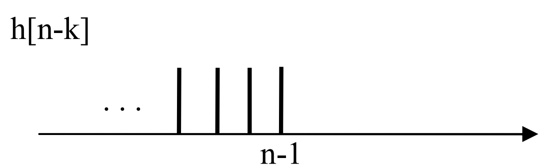
7S. Compute the convolution y[n]=x[n]*h[n] of the following pair of signals:

$$x[n] = a^n u[n-4]$$

$$h[n] = b^n u[n-1]$$ **where 0 < a < b**

Solution:





$$x[n] = a^n u[n-4]$$
$$h[n] = b^n u[n-1]$$
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$y[n] = 0 \quad for \quad n-1 < 4 \Rightarrow n < 5$$
$$y[n] = \sum_{k=4}^{n-1} a^k \beta^{n-k} \quad for\ n \geq 5$$
$$y[n] = \beta^n \sum_{k=0}^{n-5} (a/\beta)^k \quad for\ n \geq 0$$

7U. Compute the convolution y[n]=x[n]*h[n] of the following pair of signals:

$$x[n] = (\frac{1}{3})^n u[n-2]$$
$$h[n] = (\frac{1}{2})^n u[n+3]$$

Solution:

8S. The following are the impulse response of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) h[n] = $(1/5)^n$ u[n]
b) h[n] = $(0.8)^n$ u[n + 2]
c) h[n] = $(1/2)^n$ u[-n]
d) h[n] = $(5)^n$ u[3 – n]
e) h[n] = $(-1/2)^n$ u[n] + $(1.01)^n$ u[n – 1]
f) h[n] = $(-1/2)^n$ u[n] + $(1.01)^n$ u[1 – n]
g) h[n] = n $(1/3)^n$ u[n – 1]

Solution:

Notes → "System is causal if h[n]=0 for n<0 --- System is stable if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$"

a) h[n] = $(1/5)^n$ u[n]

h[n]=0 for n<0 therefore system is causal

$\sum_{n=0}^{\infty} (1/5)^n = \frac{1}{1-(1/5)} = 5/4 < \infty$ therefore the System is stable

b) h[n] = $(0.8)^n$ u[n + 2]

h[n]≠0 for n<0 therefore system is non-causal

$\sum_{n=-2}^{\infty} (0.8)^n = \sum_{n=0}^{\infty} (0.8)^{n-2} = \frac{(0.8)^{-2}}{1-(0.8)} < \infty$ therefore the System is stable

c) h[n] = $(1/2)^n$ u[-n]

h[n]≠0 for n<0 therefore system is non-causal

$\sum_{n=-\infty}^{0} (1/2)^n = \infty$ therefore the System is not stable

d) h[n] = $(5)^n$ u[3 – n]

h[n]≠0 for n<0 therefore system is non-causal

$$\sum_{n=-\infty}^{3}(5)^{n} = \sum_{n=0}^{\infty}(1/5)^{n} - \sum_{n=-3}^{0}(1/5)^{n} = \frac{1}{1-(1/5)} - (1/5)^{-3} - (1/5)^{-2} - (1/5)^{-1} - (1/5)^{0} = 625/4 < \infty$$

therefore the System is stable

e) **h[n] = (-1/2)^n u[n] + (1.01)^n u[n – 1]**

h[n]=0 for n<0 therefore system is causal

$$\sum_{n=0}^{\infty}|\,(-1/2)^{n}\,| + \sum_{n=1}^{\infty}(1.01)^{n} = \frac{1}{1-(1/2)} + \infty = \infty$$ therefore the System is not stable

f) **h[n] = (-1/2)^n u[n] + (1.01)^n u[1 – n]**

h[n]≠0 for n<0 therefore system is non-causal

$$\sum_{n=0}^{\infty}|\,(-1/2)^{n}\,| + \sum_{n=-\infty}^{1}(1.01)^{n} = \frac{1}{1-(1/2)} + \sum_{n=0}^{\infty}(1.01)^{-n} - 1.01 = 2/3 + \frac{1}{1-(1/1.01)} - 1.01 < \infty$$

therefore the System is stable

g) **n (1/3)^n u[n – 1]**

h[n]=0 for n<0 therefore system is causal

$$\sum_{n=1}^{\infty} n(1/3)^{n} = \frac{1}{1-(1/3)} = 3/4 = 0.75 < \infty$$ therefore the System is stable

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8U. The following are the impulse response of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) **h[n] = (1/5)^n u[-n]**
b) **h[n] = (0.8)^n u[n - 2]**
c) **h[n] = (1/2)^n u[n]**
d) **h[n] = (5)^n u[3 + n]**
e) **h[n] = n (1/3)^n u[n + 1]**

Solution:

9S. The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) h(t) = e $^{-4t}$ u(t – 2)
b) h(t) = e $^{-6t}$ u(3 – t)
c) h(t) = e $^{-2t}$ u(t + 50)
d) h(t) = e ^{2t} u(–1 – t)
e) h(t) = e $^{-6|t|}$
f) h(t) = t e $^{–t}$ u(t)
g) h(t) = (2 e $^{–t}$ – e $^{(t – 1000)/100}$) u(t)

Notes → "System is causal if h(t)=0 for t<0 --- System is stable if $\int_{-\infty}^{-\infty}|\,h(t)\,|\,dt$"

Solution:

a) h(t) = e $^{-4t}$ u(t – 2)

h(t)=0 for t<0 therefore System is causal

$$\int_{2}^{\infty}|\,e^{-4t}\,|\,dt = -1/4(e^{-\infty} - e^{-8}) = 1/4(e^{-8}) < \infty$$ therefore System is stable

b) h(t) = e $^{-6t}$ u(3 – t)

h(t)≠0 for t<0 therefore System is not causal

$$\int_{-\infty}^{3}|\,e^{-6t}\,|\,dt = 1/6(e^{-18} - e^{\infty}) = \infty$$ therefore System is unstable

c) h(t) = e $^{-2t}$ u(t + 50)

h(t)≠0 for t<0 therefore System is not causal

$$\int_{-50}^{\infty} | e^{-2t} | dt = -1/2(e^{-\infty} - e^{100}) = 1/2e^{100} < \infty$$ therefore System is stable

d) h(t) = e ^{2t} u(–1 – t)

h(t)≠0 for t<0 therefore System is not causal

$$\int_{-\infty}^{-1} | e^{2t} | dt = 1/2(e^{-2} - e^{-\infty}) = e^{-2}/2 < \infty$$ therefore System is stable

e) h(t) = e $^{-6|t|}$

h(t) ≠0 for t<0 therefore System is not causal

$$2\int_{0}^{\infty} | e^{-6|t|} | dt = -1/3(e^{-\infty} - e^{0}) = 1/3 < \infty$$ therefore System is stable

f) h(t) = t e $^{-t}$ u(t)

h(t) =0 for t<0 therefore System is causal

$$\int_{0}^{\infty} | te^{-t} | dt = 1 < \infty$$ therefore System is stable (Use integral by part to solve)

g) h(t) = (2 e $^{-t}$ – e $^{(t-1000)/100}$) u(t)

h(t) =0 for t<0 therefore System is causal

$$\int_{0}^{\infty} | (2e^{-t} - e^{(t-1000)/100}) | dt = \infty$$ therefore System is unstable

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9U. The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.

a) h(t) = e $^{-3t}$ u(t + 4)
b) h(t) = e $^{-5t}$ u(13 – t)
c) h(t) = e $^{-2t}$ u(-t + 10)
d) h(t) = e ^{2t} u(–3 – t)
e) h(t) = e$^{-10|t|}$

Solution:

10S. h[n] = (-1/2)^n u[n] + (1.01)^n u[n – 1] is the impulse of response of a system. Determine if the system is Causal and/or Stable

Solution:

h[n] = (-1/2)^n u[n] + (1.01)^n u[n – 1]

Notes → "System is causal if h[n]=0 for n<0 --- System is stable if $\sum_{n=\infty}^{\infty} | h[n] | < \infty$"

h[n]=0 for n<0 therefore system is causal

$$\sum_{n=0}^{\infty} | (-1/2)^n | + \sum_{n=1}^{\infty} (1.01)^n = \frac{1}{1-(1/2)} + \infty = \infty$$ therefore the System is not stable

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10U. h[n] = (-1/8)^n u[n + 1] + (1.21)^n u[n] is the impulse of response of a system. Determine if the system is Causal and/or Stable