

Signals & Systems - Chapter 3

1S. Continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T=8$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_1 = a_{-1} = 2, \quad a_3 = a_{-3} = 4j.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk w_0 t} = 2e^{-j w_0 t} + 2e^{j w_0 t} - 4j e^{-j 3 w_0 t} + 4j e^{j 3 w_0 t}$$

Apply Eulers

$$x(t) = 2(2 \cos w_0 t) + 4j(2j \sin 3w_0 t) = 4 \cos w_0 t - 8 \sin 3w_0 t$$

$$w_0 = 2\pi / T = 2\pi / 8 = \pi / 4$$

$$x(t) = 4 \cos \pi t / 4 - 8 \cos(3\pi t / 4 - \pi / 2)$$

1U. Continuous-time periodic signal $x(t)$ is real valued and has a fundamental period $T=12$. The nonzero Fourier series coefficients for $x(t)$ are

$$a_0 = 4, \quad a_2 = a_{-2} = j, \quad a_3 = a_{-3} = -4j.$$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k).$$

Solution:

2S. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N=5$. The nonzero Fourier series coefficient for $x[n]$ are

$$a_0 = 1, \quad a_2 = a_{-2} = e^{j\pi/4}, \quad a_4 = a_{-4} = 2e^{j\pi/3}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k).$$

Solution:

Note: $a_k = a_{k+rN}$ where r is any integer therefore

$$a_{-1} = a_{(-1+5)} = a_4 = 2e^{j\pi/3} \quad \text{and} \quad a_1 = a_{(1-5)} = a_{-4} = 2e^{-j\pi/3}$$

$$x[n] = \sum_{k=-2}^2 a_k e^{jk \omega_0 n} = a_0 + a_{-1} e^{-j\omega_0 n} + a_1 e^{j\omega_0 n} + a_{-2} e^{-j2\omega_0 n} + a_2 e^{j2\omega_0 n}$$

$$x[n] = 1 + 2e^{j\frac{\pi}{3}} e^{-j\omega_0 n} + 2e^{-j\frac{\pi}{3}} e^{j\omega_0 n} + e^{-j\frac{\pi}{4}} e^{-j2\omega_0 n} + e^{j\frac{\pi}{4}} e^{j2\omega_0 n}$$

$$x[n] = 1 + 2 \left\{ e^{j(\frac{\pi}{3} - \omega_0 n)} + e^{-j(\frac{\pi}{3} - \omega_0 n)} \right\} + \left\{ e^{-j(\frac{\pi}{4} + 2\omega_0 n)} + e^{j(\frac{\pi}{4} + 2\omega_0 n)} \right\}$$

Apply Eulers

$$x[n] = 1 + 4 \cos \left(-\omega_0 n + \frac{\pi}{3} \right) + 2 \cos \left(2\omega_0 n + \frac{\pi}{4} \right)$$

$$w_0 = \frac{2\pi}{N} = \frac{2\pi}{5}$$

$$x[n] = 1 + 4 \sin \left(-\omega_0 n + \frac{5\pi}{6} \right) + 2 \sin \left(2\omega_0 n + \frac{3\pi}{4} \right)$$

2U. A discrete-time periodic signal $x[n]$ is real valued and has a fundamental period $N=9$. The nonzero Fourier series coefficient for $x[n]$ are

$$a_0 = 2, \quad a_3 = a_{-3}^* = 3e^{j\pi/2}, \quad a_4 = a_{-4}^* = 2e^{-j\pi/4}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(\omega_k n + \phi_k).$$

Solution:

3S. For the continuous-time period signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Solution:

Apply Eulers to the $x(t)$

$$x(t) = 2 + \frac{1}{2} \left[e^{+j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t} \right] - \frac{1}{2j} \left[e^{+j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t} \right]$$

$$x(t) = 2 + \frac{1}{2} e^{+j2(\frac{2\pi}{6})t} + \frac{1}{2} e^{-j2(\frac{2\pi}{6})t} - \frac{1}{2j} e^{+j5(\frac{2\pi}{6})t} + \frac{1}{2j} e^{-j5(\frac{2\pi}{6})t}$$

$$\omega_0 = \frac{2\pi}{6} \text{ Fundamental Frequency}$$

$$a_0=2; \quad a_2=a_{-2}=1/2; \quad a_5=a_{-5}^*=-1/(2j);$$

3U. For the continuous-time period signal

$$x(t) = 3 + 4\cos\left(\frac{4\pi}{7}t\right) + 2\sin\left(\frac{3\pi}{5}t\right)$$

determine the fundamental frequency ω_0 and the Fourier series coefficients a_k .

Solution:

4S. Use the Fourier series analysis equation to calculate the coefficients a_k for the continuous-time periodic signal

$$x(t) = \begin{cases} 1.5 & \text{for } 0 \leq t < 1 \\ -1.5 & \text{for } 1 \leq t < 2 \end{cases}$$

with fundamental frequency $\omega_0 = \pi$.

Solution: Continuous-time system, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \text{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \quad \text{Fourier Series Analysis Equation}$$

$$w_0 = \pi \rightarrow \pi T = 2\pi \rightarrow T = 2$$

$$a_k = \frac{1}{2} \int_0^2 x(t) e^{-jk\omega_0 t} dt = \frac{1}{2} \int_0^1 1.5 e^{-jk\omega_0 t} dt - \frac{1}{2} \int_1^2 1.5 e^{-jk\omega_0 t} dt$$

$$\text{for } k = 0 \rightarrow a_0 = 0$$

$$\text{for } k \neq 0$$

$$a_k = \frac{1.5}{-2jk\omega_0} (e^{-jk\omega_0} - 1 - e^{-j2k\omega_0} + e^{-jk\omega_0}) = \frac{1.5}{-2jk\pi} (2e^{-jk\pi} - 1 - e^{-j2k\pi}) = \frac{1.5}{-2jk\pi} (2e^{-jk\pi} - 2)$$

$$\text{for even } k \rightarrow a_k = 0$$

$$\text{for odd } k \rightarrow a_k = \frac{1.5}{-2jk\pi} (2\cos k\pi - j2\sin k\pi - 2) = \frac{3}{jk\pi}$$

4U. Use the Fourier series analysis equation to calculate the coefficients a_k for the continuous-time periodic signal

$$x(t) = \begin{cases} +1 & \text{for } 0 \leq t < 0.5 \text{ m sec} \\ -1 & \text{for } 0.5 \text{ m sec} \leq t < 1 \text{ m sec} \end{cases}$$

with fundamental frequency $\omega_0 = 2000\pi$.

Solution:

5S. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$x_1(t) = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^k e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$x_2(t) = \sum_{k=-100}^{100} \cos(k\pi) e^{jk\left(\frac{2\pi}{50}\right)t}$$

$$x_3(t) = \sum_{k=-100}^{100} j \sin\left(\frac{k\pi}{2}\right) e^{jk\left(\frac{2\pi}{50}\right)t}$$

Use Fourier series properties to help answer the following questions:

- Which of the three signals is/are real valued?
- Which of the three signals is/are even?

Solution:

- if $a_k = a_{-k}^*$ Then the signal $x(t)$ is real otherwise it is not.

Note: conjugate means that $(A+jB)^* = (A-jB)$

$x_1(t)$ Fourier series coefficients are $a_k = (1/2)^k$ for $0 \leq k \leq 100$ otherwise $a_k = 0$
 We know that $\{a_k = (1/2)^k\} \neq \{a_{-k}^* = (1/2)^{-k}\}$ therefore $x_1(t)$ is not Real

$x_2(t)$ Fourier series coefficients are $a_k = \cos(k\pi)$ for $-100 \leq k \leq 100$ otherwise $a_k = 0$
 We know that $\{a_k = \cos(k\pi)\} = \{a_{-k}^* = \cos(-k\pi)\}$ therefore $x_2(t)$ is Real

$x_3(t)$ Fourier series coefficients are $a_k = j \sin(k\pi/2)$ for $-100 \leq k \leq 100$ otherwise $a_k = 0$
 We know that $\{a_k = j \sin(k\pi/2)\} = \{a_{-k}^* = -j \sin(-k\pi/2)\}$ therefore $x_3(t)$ is Real

- For a signal $x(t)$ to be even its Fourier Series Coefficient a_k must be even
 In other words the relationship " $x(t) = x(-t) \leftrightarrow a_k = a_{-k}$ " is true

Which means only $x_2(t)$ is even since only for this function $a_k = a_{-k}$

5U. Consider three continuous-time periodic signals whose Fourier series representations are as follows:

$$\begin{aligned} x_1(t) &= \sin(2000k\pi t) \quad \text{where } k \text{ is integer} \\ x_2(t) &= 10\cos(15.294k\pi t) + 10 \quad \text{where } k \text{ is integer} \\ x_3(t) &= \sin(2000k\pi t) + j15 \quad \text{where } k \text{ is integer} \end{aligned}$$

Which of the above signals have a Fourier Series Coefficient (a_k) that are:

- a) Even
- b) Odd
- c) Pure Imaginary
- d) Real

Solution:

6S. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}.$$

Solution: For Discrete-time system, we have

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \quad \text{Fourier Series Synthesis Equation}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \quad \text{Fourier Series Analysis Equation}$$

a)

first understand the signal $x[n] = \sum_{m=-\infty}^{\infty} \{4\delta[n-4m] + 8\delta[n-1-4m]\}$

using the definition of impulse function we can write:

$$\begin{aligned} x[n] &= 4 \quad \text{for } n=4m \\ &= 8 \quad \text{for } n=4m+1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

We see that the signal is periodic with a fundamental period of $N=4$.

[If you don't see it, just find value for $x[0] = 4$, $x[1] = 8$, $x[2] = 0$, $x[3] = 0$, $x[4] = 4$, $x[5] = 8$, ... which repeats every four terms]

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{4} \sum_{n=0}^3 x[n] e^{-jk(2\pi/4)n} = \frac{1}{4} [4e^0 + 8e^{-jk(2\pi/4)}] = 1 + 2e^{-jk(\pi/2)}$$

Therefore:

$$a_0 = 3, \quad a_1 = 1 - 2j, \quad a_2 = -1, \quad a_3 = 1 + 2j$$

6U. Use the analysis equation to evaluate the numerical values of one period of the Fourier series coefficients of the periodic signal

$$x[n] = \sum_{m=-\infty}^{\infty} \{3\delta[n-5m] + 2\delta[n-2-5m]\}.$$

Solution:

7S. Let $x[n]$ be a real and odd periodic signal with period $N=7$ and Fourier coefficient a_k . Given that

$$a_{15} = j, \quad a_{16} = 2j, \quad a_{17} = 3j.$$

Determine the values of a_0 , a_{-1} , a_{-2} and a_{-3} .

Solution:

Using the properties of Fourier Series we could state:

1) Period with period $N=7 \rightarrow a_k = a_{k+7n}$

$$a_1 = a_{1+2 \cdot 7} = a_{15} = j$$

$$a_2 = a_{2+2 \cdot 7} = a_{16} = 2j$$

$$a_3 = a_{3+2 \cdot 7} = a_{17} = 3j$$

2) real and odd $x(t) \rightarrow a_k$ is purely imaginary and odd ($a_k = -a_{-k}$)

$$a_0 = 0$$

$$a_{-1} = -a_1 = -j$$

$$a_{-2} = -a_2 = -2j$$

$$a_{-3} = -a_3 = -3j$$

7U. Let $x[n]$ be a real and odd periodic signal with period $N=9$ and Fourier coefficient a_k . Given that

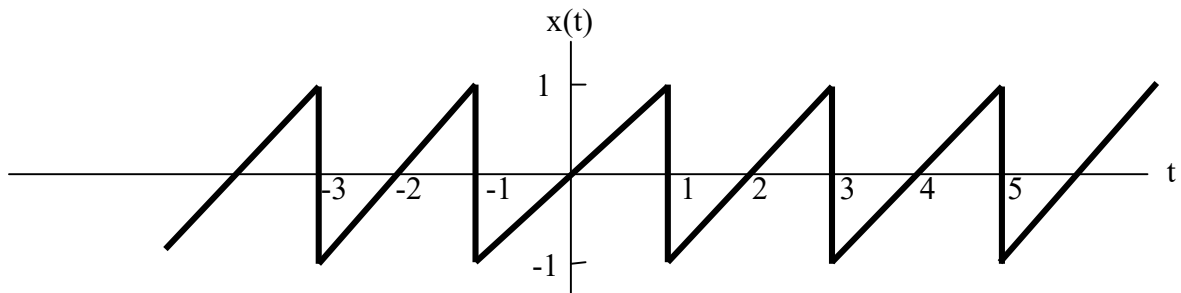
$$a_{15} = 2j, \quad a_{16} = 3j, \quad a_{17} = -4j.$$

Determine the values of a_0 , a_{-1} , a_{-2} and a_{-3} .

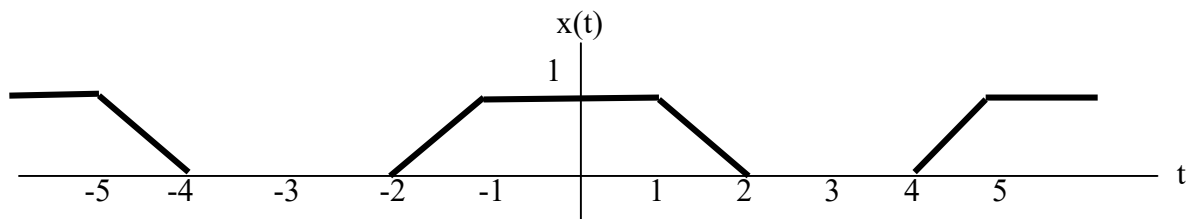
Solution:

8S. Determine the Fourier series representations for the following signals:

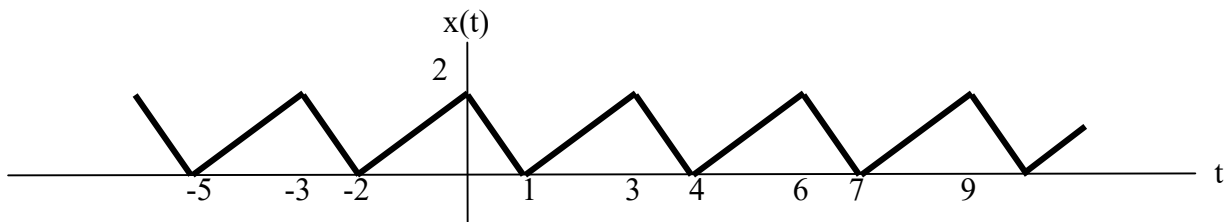
a) Each $x(t)$ illustrated in the following figure



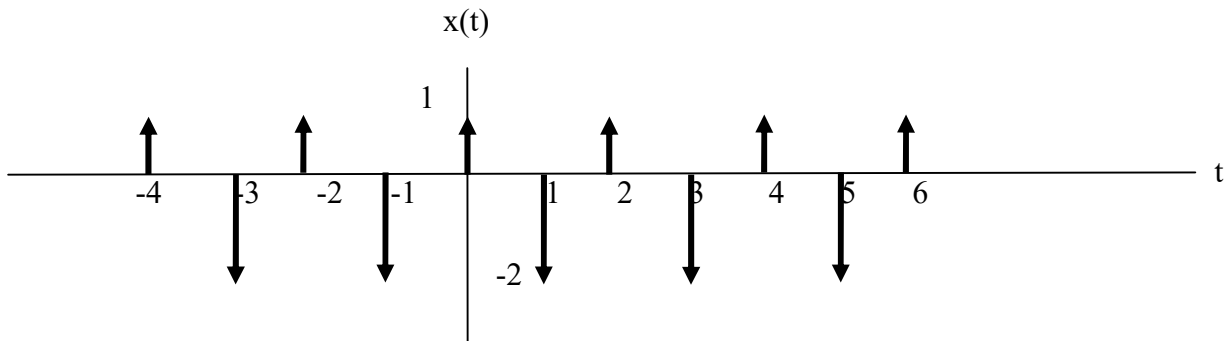
b) Each $x(t)$ illustrated in the following figure



c) Each $x(t)$ illustrated in the following figure



d) Each $x(t)$ illustrated in the following figure



e) $x(t)$ is a periodic signal with period 2 and
 $x(t) = e^{-t}$ for $-1 < t < 1$

Solution:

a)

$$T = 2 \rightarrow \omega_0 = 2\pi / T = \pi$$

$$x(t) = t$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{2} \int_{-1}^1 t e^{-jk\pi t} dt$$

$$a_0 = 0$$

$$\text{Integration by part} \rightarrow \int u dv = uv - \int v du$$

$$a_k = \frac{j(-1)^k}{k\pi} \quad \text{for } k \neq 0$$

b)

$$T = 6 \rightarrow w_0 = 2\pi / T = \pi / 3$$

$$x(t) = t + 2 \quad \text{for } -2 < t < -1$$

$$1 \quad \text{for } -1 < t < +1$$

$$2 - t \quad \text{for } +1 < t < +2$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$$

$$a_k = \frac{1}{6} \left[\int_{-2}^{-1} (t+2) e^{-jkw_0 t} dt + \int_{-1}^1 e^{-jkw_0 t} dt + \int_1^2 (2-t) e^{-jkw_0 t} dt \right]$$

$$a_0 = 1/2$$

$$a_k = 0 \quad \text{for even } k$$

$$= \frac{6}{\pi^2 k^2} \sin\left(\frac{\pi k}{2}\right) \sin\left(\frac{\pi k}{6}\right) \quad \text{for odd } k$$

c)

$$T = 3 \rightarrow w_0 = 2\pi / T = 2\pi / 3$$

$$x(t) = t + 2 \quad \text{for } -2 < t < 0$$

$$2 - 2t \quad \text{for } 0 < t < +1$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$$

$$a_k = \frac{1}{3} \left[\int_{-2}^0 (t+2) e^{-jkw_0 t} dt + \int_0^1 (2-2t) e^{-jkw_0 t} dt \right]$$

$$a_0 = 1$$

$$a_k = \frac{3j}{2\pi^2 k^2} \left[e^{jk2\pi/3} \sin(k2\pi/3) + 2e^{jk\pi/3} \sin(k\pi/3) \right] \quad \text{for } k \neq 0$$

d)

$$T = 2 \rightarrow w_0 = 2\pi / T = \pi$$

$$x(t) = 1 \quad \text{for } t = 0$$

$$-2 \quad \text{for } t = 1$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$$

$$a_0 = -1/2$$

$$a_k = \frac{1}{2} - (-1)^k \quad \text{for } k \neq 0$$

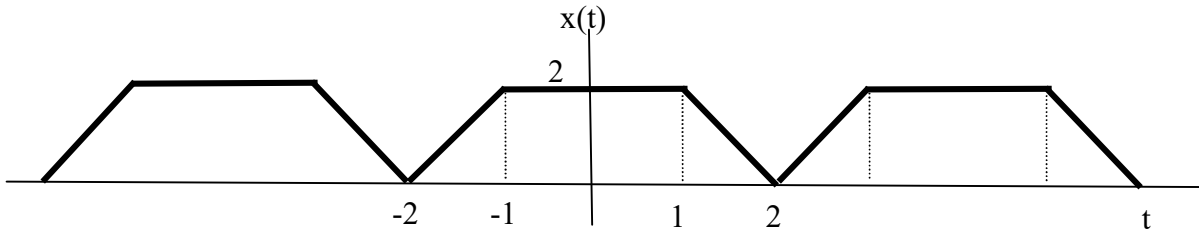
e)

$$T = 2 \rightarrow w_0 = 2\pi/T = \pi$$

$$x(t) = e^{-t} \text{ for } -1 < t < 1$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \int_{-1}^1 e^{-t} e^{-jk\pi t} dt = \frac{-1}{1+jk\pi} (e^{-(1+jk\pi)t}) \Big|_{-1}^1 = \frac{e^{(1+jk\pi)} - e^{-(1+jk\pi)}}{1+jk\pi} \text{ for all } k$$

8U. Determine the Fourier Series representation for the signal shown in the following figure:



Solution:

9S. A discrete-time periodic signal $x[n]$ is real valued and has fundamental period $N=5$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \quad a_2 = a_{-2}^* = 2e^{j\pi/6}, \quad a_4 = a_{-4}^* = e^{j\pi/3}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k).$$

Solution:

$$\begin{aligned} w_0 &= 2\pi/N = 2\pi/5 \\ x[n] &= \sum_{k=-2}^2 a_k e^{jk\omega_0 n} = a_{-2} e^{-j2(2\pi/5)n} + a_{-1} e^{j(2\pi/5)n} + a_0 + a_1 e^{j(2\pi/5)n} + a_2 e^{j2(2\pi/5)n} \\ x[n] &= 2e^{-\frac{j\pi}{5}} e^{-j(\frac{4\pi}{5})n} + e^{\frac{j\pi}{5}} e^{-j(\frac{2\pi}{5})n} + 2 + e^{-\frac{j\pi}{5}} e^{j(\frac{2\pi}{5})n} + 2e^{\frac{j\pi}{5}} e^{j(\frac{4\pi}{5})n} \\ &\text{Apply Euler Identity} \\ x[n] &= 2 + 4\cos[(4\pi/5)n + \pi/6] + 2\cos[(\frac{2\pi}{5})n - \pi/3] \\ &\text{add } \pi/2 \\ x[n] &= 2 + 4\sin[(4\pi/5)n + 4\pi/6] + 2\sin[(2\pi/5)n + \pi/6] \end{aligned}$$

9U. A discrete-time periodic signal $x[n]$ is real valued and has fundamental period $N=7$. The nonzero Fourier series coefficients for $x[n]$ are

$$a_0 = 2, \quad a_1 = a_{-1}^* = e^{j\pi/3}, \quad a_3 = a_{-3}^* = 3e^{j\pi}$$

Express $x[n]$ in the form

$$x[n] = A_0 + \sum_{k=1}^{\infty} A_k \sin(w_k n + \phi_k).$$

Solution:

10S. Determine the Fourier series coefficients for the following discrete-time periodic signal. Plot the magnitude and phase of each set of coefficients a_k .

$$x[n] = \sin(2\pi n/3) \cos(\pi n/2)$$

Solution:

first term $w_0 = 2\pi/N = 2\pi/3 \Rightarrow N_1 = 3$

second term $w_0 = 2\pi/N = \pi/2 \Rightarrow N_2 = 4$

Therefore period for $X[n] \Rightarrow N = 12$

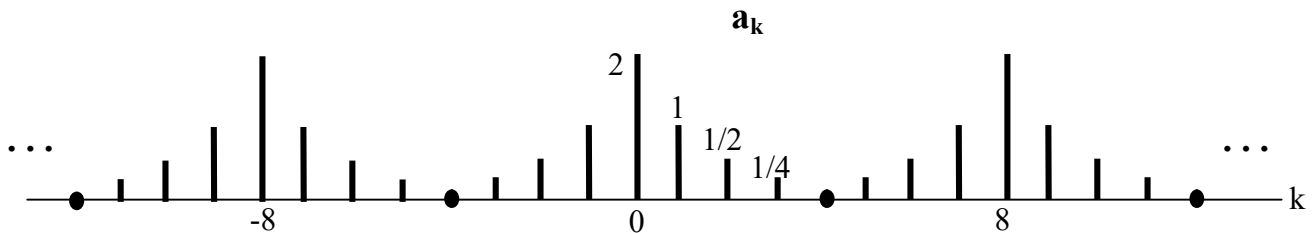
$$a_k = \frac{1}{N} \sum_{n=0}^{11} x[n] e^{-jkw_0 n} = \frac{1}{12} \sum_{n=0}^{11} \sin(2\pi n/3) \cos(\pi n/2) e^{-jk \omega_0 n}$$

10U. Determine the Fourier series coefficients for the following discrete-time periodic signal. Plot the magnitude and phase of each set of coefficients a_k .

$$x[n] = \sin(6\pi n/7) e^{j\pi n/10}$$

Solution:

11S. The signal represented by the following Fourier series coefficients is a periodic with period 8. Determine the signal $x[n]$.



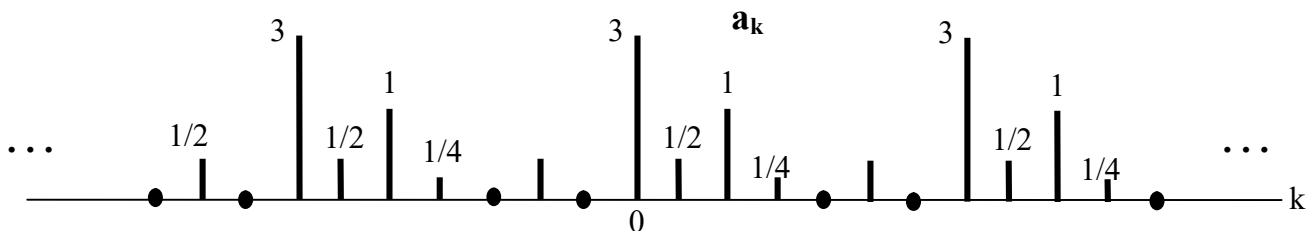
Solution:

$$w_0 = 2\pi/8 = \pi/4$$

$$x[n] = \sum_{k=-4}^3 a_k e^{jkw_0 n} = 2 + e^{j(\pi/4)n} + e^{-j(\pi/4)n} + \frac{1}{2} e^{j(\pi/2)n} + \frac{1}{2} e^{-j(\pi/2)n} + \frac{1}{4} e^{j(3\pi/4)n} + \frac{1}{4} e^{-j(3\pi/4)n} + 0$$

$$x[n] = 2 + 2 \cos(\pi n/4) + \cos(\pi n/2) + (1/2) \cos(3\pi n/4)$$

11U. The signal represented by the following Fourier series coefficients is a periodic with period 7. Determine the signal $x[n]$.



Solution: