

Signals & Systems - Chapter 5

1S. Use the Fourier transform analysis equation to calculate the Fourier transforms of:

a) $(\frac{1}{2})^{n-1}u[n-1]$

b) $(\frac{1}{2})^{|n-1|}$

Solution: For discrete-time Fourier transform, we have:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw \quad \text{Synthesis equation}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \quad \text{Analysis equation}$$

a)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \quad \text{Analysis equation}$$

$$X(e^{jw}) = \sum_{n=1}^{+\infty} (\frac{1}{2})^{n-1} e^{-jwn} = \sum_{n=0}^{+\infty} (\frac{1}{2})^n e^{-jw(n+1)} = e^{-jw} \sum_{n=0}^{+\infty} \left(\frac{1}{2} e^{-jw}\right)^n$$

Apply Infinite Sum

$$X(e^{jw}) = e^{-jw} \frac{1}{1 - (\frac{1}{2} e^{-jw})}$$

Note

We have Finite sum $\sum_{n=0}^N a^n = \frac{1+a^{N+1}}{1-a}$ and if $|a| < 1$ & $N \rightarrow \infty$ then we have infinite sum $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

b)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \quad \text{Analysis equation}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-(n-1)} e^{-jwn} + \sum_{n=2}^{+\infty} (\frac{1}{2})^{(n-1)} e^{-jwn} = \sum_{n=0}^{\infty} (\frac{1}{2})^n e^{-jw(-n+1)} + \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} e^{-jw(n+2)}$$

$$X(e^{jw}) = (e^{jw}) \sum_{n=0}^{+\infty} \left(\frac{1}{2} e^{jw}\right)^n + (\frac{1}{2} e^{-j2w}) \sum_{n=0}^{+\infty} \left(\frac{1}{2} e^{-jw}\right)^n$$

Apply Finite Sum

$$X(e^{jw}) = \frac{e^{jw}}{1 - \frac{1}{2} e^{jw}} + \frac{\frac{1}{2} e^{-j2w}}{1 - \frac{1}{2} e^{-jw}}$$

1U. Use the Fourier transform analysis equation to calculate the Fourier transforms of:

a) $(\frac{1}{2})^{n+1}u[n+1]$

b) $(\frac{1}{2})^{|n-5|}$

Solution:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw \quad \text{Synthesis equation}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \quad \text{Analysis equation}$$

2S. Use the Fourier transform synthesis equation to determine the inverse Fourier transform of:

a) $X_1(e^{jw}) = \sum_{k=-\infty}^{\infty} \{4\pi\delta(w - 4\pi k) + 2\pi\delta(w - \frac{\pi}{2} - 3\pi k) + 2\pi\delta(w + \frac{\pi}{2} - 3\pi k)\}$

b) $X_2(e^{jw}) = \begin{cases} 2j & \text{for } 0 < w \leq \pi \\ -2j & \text{for } -\pi < w \leq 0 \end{cases}$

Solution:

a)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw \quad \text{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \{4\pi\delta(w - 4\pi k) + 2\pi\delta(w - \frac{\pi}{2} - 3\pi k) + 2\pi\delta(w + \frac{\pi}{2} - 3\pi k)\} e^{jwn} dw$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{4\pi\delta(w) + 2\pi\delta(w - \frac{\pi}{2}) + 2\pi\delta(w + \frac{\pi}{2})\} e^{jwn} dw$$

$$x[n] = 2e^{j0} + e^{j(\pi/2)n} + e^{-j(\pi/2)n} = 2 + 2\cos(\pi n / 2)$$

b)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw \quad \text{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi} \int_0^{\pi} 2je^{jwn} dw - \frac{1}{2\pi} \int_{-\pi}^0 2je^{jwn} dw = (j/\pi) \left[-\frac{1 - e^{jn\pi}}{jn} + \frac{e^{-jn\pi} - 1}{jn} \right]$$

$$x[n] = \frac{2}{n\pi} [-1 + \cos(n\pi)] = -(4/(n\pi)) \sin^2(n\pi / 2)$$

2U. Use the Fourier transform synthesis equation to determine the inverse Fourier transform of:

a) $X_1(e^{jw}) = \sum_{k=-\infty}^{\infty} \{2\pi\delta(w - 2\pi k) + \pi\delta(w - \frac{\pi}{3} - 2\pi k) + \pi\delta(w + \frac{\pi}{4} - 5\pi k)\}$

$$X_2(e^{jw}) = \begin{cases} 5j & \text{for } 0 < w \leq 2\pi/3 \\ -5j & \text{for } -2\pi/3 < w \leq 0 \\ 0 & \text{Otherwise} \end{cases}$$

b) $X_2(e^{jw}) = \begin{cases} 5j & \text{for } 0 < w \leq 2\pi/3 \\ -5j & \text{for } -2\pi/3 < w \leq 0 \\ 0 & \text{Otherwise} \end{cases}$

Solution:

3S. Given that $x[n]$ has Fourier transform $X(e^{jw})$, express the Fourier transform of the following signals in terms of $X(e^{jw})$. You may use the Fourier transform properties table.

a) $x_1[n] = x[1 - n] + x[-1 - n]$

b) $x_2[n] = \frac{x^*[-n] + x[n]}{2}$

c) $x_3[n] = (n - 1)^2 x[n]$

Solution:

a)

$$\begin{aligned} \text{Time Reversal: } x[-n] &\xrightarrow{FT} X(e^{-j\omega}) \\ \text{Time Shift: } x[n - n_0] &\xrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega}) \\ \text{Therefore} \end{aligned}$$

b) $x[n] = x[1 - n] + x[-1 - n] \xrightarrow{FT} e^{j\omega} X(e^{-j\omega}) + e^{-j\omega} X(e^{-j\omega}) = 2X(e^{-j\omega}) \cos \omega$

Time Reversal: $x[-n] \xrightarrow{FT} X(e^{-j\omega})$

Conjugation Property: $x^*[n] \xrightarrow{FT} X^*(e^{-j\omega})$

Therefore

$$x[n] = (1/2) \{x^*[-n] + x[n]\} \xrightarrow{FT} (1/2) \{X^*(e^{j\omega}) + X(e^{j\omega})\} = \text{Real}\{X(e^{j\omega})\}$$

c)

Differentiation in Frequency: $nx[n] \xrightarrow{FT} j \frac{dX(e^{j\omega})}{d\omega}$

Differentiation in Frequency Again: $n^2 x[n] \xrightarrow{FT} -\frac{d^2 X(e^{j\omega})}{d\omega^2}$

Therefore

$$x[n] = (n - 1)^2 x[n] = (n^2 - 2n + 1)x[n] \xrightarrow{FT} -\frac{d^2 X(e^{j\omega})}{d\omega^2} - 2j \frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

3U. Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transform of the following signals in terms of $X(e^{j\omega})$. You may use the Fourier transform properties table.

a) $x_1[n] = x[1 + n] + x[1 - n]$

b) $x_2[n] = \frac{2x[-n] + 4x^*[n]}{3}$

c) $x_3[n] = (n + 3)^2 x[n - 1]$

Solution:

4S. For each of the following Fourier transforms, use Fourier transform properties table to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

a) $X(e^{j\omega}) = e^{-j\omega} \sum_{k=1}^{10} (\sin k\omega)$

b) $X(e^{j\omega}) = j \sin(\omega) \cos(5\omega)$

c)

$$X(e^{jw}) = A(w) + e^{+jB(w)}$$

$$\text{where } B(w) = -\frac{3w}{2} + \pi \text{ and } A(w) = 1 \text{ for } 0 \leq |w| \leq \frac{\pi}{8}$$

$$0 \text{ for } \frac{\pi}{8} < |w| \leq \pi$$

Solution:

Relevant Properties (Source Properties Table)

- 1) $X(e^{jw})$ real and even $\rightarrow x[n]$ real and even
- 2) $X(e^{jw})$ purely imaginary and Odd $\rightarrow x[n]$ real and odd
- 3) $X(e^{jw})$ pure imaginary $\rightarrow x[n]$ real
- 4) $X(e^{jw})$ pure real $\rightarrow x[n]$ real

a)

We have property #2 therefore the reverse is true also meaning

$$2') x_1[n] \text{ real and odd} \rightarrow X_1(e^{jw}) \text{ purely imaginary and Odd}$$

We see that $X_1(e^{jw}) = \sum_{k=1}^{10} (\sin kw)$ is real and odd therefore Corresponding signal $x_1[n]$ is purely imaginary and odd.

$$x[n] = x_1[n-1] \text{ to count for } e^{-jw}$$

therefore $x[n]$ is also purely imaginary but it is neither odd or even.

b)

Since $X(e^{jw})$ is purely imaginary and odd $\rightarrow x[n]$ is real and odd (Property #2)

c)

Since $X(e^{jw}) = X^*(e^{-jw}) \rightarrow x[n]$ is real but neither odd or even.

4U. For each of the following Fourier transforms, use Fourier transform properties table to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

a) $X(e^{jw}) = e^{-2jw} \sum_{k=2}^{15} (\sin kw)$

b) $X(e^{jw}) = j \sin(3w) \cos(2w)$

c)

$$X(e^{jw}) = A(w) + e^{+jB(w)}$$

$$\text{where } B(w) = +\frac{5w}{2} - \pi \text{ and } A(w) = 1 \text{ for } 0 \leq |w| \leq \frac{\pi}{8}$$

$$0 \text{ for } \frac{\pi}{8} < |w| \leq \pi$$

Solution:

5S. Compute the Fourier transform of each of the following signals:

a) $x[n] = u[n-2] - u[n-6]$

b) $x[n] = (1/3)^{|n|} u[-n-2]$

- c) $x[n] = (1/2)^{|n|} \cos\{(\pi/8)(n-1)\}$
 d) $x[n] = \sin(n\pi/2) + \cos(n)$
 e) $x[n] = \left(\frac{\sin(\pi n/5)}{\pi n}\right) \cos\left(\frac{7\pi}{2}n\right)$

Solution:

- a) We can rewrite the signal as

$$x[n] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$$

Based on transformation table & time shift property

$$x(e^{jw}) = e^{-j2w} + e^{-j3w} + e^{-j4w} + e^{-j5w}$$

- b) Use the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{-2} \left(\frac{1}{3}\right)^{-n} e^{-jwn} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n e^{jwn} = \sum_{n=0}^{\infty} \left(\frac{1}{3}e^{jw}\right)^{n+2} = \frac{1}{9}e^{j2w} \sum_{n=0}^{\infty} \left(\frac{1}{3}e^{jw}\right)^n$$

$$\text{Finite sum } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{where } a < 1$$

$$X(e^{jw}) = \frac{e^{j2w}}{9} \frac{1}{1 - \frac{1}{3}e^{jw}}$$

- c) Use the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi(n-1)}{8}\right)e^{-jwn} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \left[e^{j\left(\frac{\pi(n-1)}{8}\right)} + e^{-j\left(\frac{\pi(n-1)}{8}\right)} \right] e^{-jwn}$$

$$X(e^{jw}) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \left[e^{-j\frac{\pi}{8}} e^{j\left(\frac{\pi}{8}-w\right)n} + e^{j\frac{\pi}{8}} e^{j\left(-\frac{\pi}{8}-w\right)n} \right] =$$

$$X(e^{jw}) = \frac{e^{-j\frac{\pi}{8}}}{2} \sum_{n=0}^{\infty} \left[\frac{1}{2} e^{j\left(-\frac{\pi}{8}-w\right)n} \right]^n + \left[\frac{1}{2} e^{j\left(\frac{\pi}{8}-w\right)n} \right]^n + \frac{e^{j\frac{\pi}{8}}}{2} \sum_{n=-\infty}^0 \left[\frac{1}{2} e^{j\left(\frac{\pi}{8}+w\right)n} \right]^n + \left[\frac{1}{2} e^{j\left(-\frac{\pi}{8}+w\right)n} \right]^n$$

$$\text{Finite sum } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{where } a < 1$$

$$X(e^{jw}) = \frac{e^{-j\frac{\pi}{8}}}{2\left(1 - \frac{1}{2}e^{j\left(-\frac{\pi}{8}-w\right)}\right)} + \frac{e^{-j\frac{\pi}{8}}}{2\left(1 - \frac{1}{2}e^{j\left(\frac{\pi}{8}-w\right)}\right)} + \frac{e^{j\frac{\pi}{8}}}{2\left(1 - \frac{1}{2}e^{j\left(\frac{\pi}{8}+w\right)}\right)} + \frac{e^{j\frac{\pi}{8}}}{2\left(1 - \frac{1}{2}e^{j\left(-\frac{\pi}{8}+w\right)}\right)}$$

- d) Use the Fourier Transform Table

$$x(n) = \sin(\pi n/2) + \cos(n) = \frac{1}{2j} [e^{j\pi n/2} - e^{-j\pi n/2}] + \frac{1}{2} [e^{jn} + e^{-jn}]$$

$$\text{From table } e^{jw_0 n} \xrightarrow{F.T.} 2\pi \sum_{l=-\infty}^{\infty} \delta(w - w_0 - 2\pi l)$$

$$X(e^{jw}) = \frac{2\pi}{2j} \left[\sum_{l=-\infty}^{\infty} \delta\left(w - \frac{\pi}{2} - 2\pi l\right) - \sum_{l=-\infty}^{\infty} \delta\left(w + \frac{\pi}{2} - 2\pi l\right) \right] + \frac{2\pi}{2} \left[\sum_{l=-\infty}^{\infty} \delta(w - 1 - 2\pi l) + \sum_{l=-\infty}^{\infty} \delta(w + 1 - 2\pi l) \right]$$

e)

$$x[n] = \left(\frac{\sin(\pi n/5)}{\pi n}\right) \cos\left(\frac{7\pi}{2}n\right) = \left(\frac{\sin(\pi n/5)}{\pi n}\right) (1/2)(e^{j\frac{7\pi}{2}n} + e^{-j\frac{7\pi}{2}n})$$

$$\text{From FT Table } \frac{\sin(\pi n/5)}{\pi n} \xrightarrow{FT} x[w] = 1 \quad \text{for } 0 \leq |w| \leq \frac{\pi}{5}$$

0 otherwise

$x[w]$ periodic with period 2π

$$\text{From FT Property Table } e^{jw_0 n} x[n] \xrightarrow{FT} X(e^{j(w-w_0)})$$

$$\text{Therefore } \frac{\sin(\pi n/5)}{\pi n} \cos\left(\frac{7\pi}{2}n\right) \xrightarrow{FT}$$

$$x[w] = \left\{ \frac{1}{2} \quad \text{for } 0 \leq |w - \frac{7\pi}{2}| \leq \frac{\pi}{5} \right\} + \left\{ \frac{1}{2} \quad \text{for } 0 \leq |w + \frac{7\pi}{2}| \leq \frac{\pi}{5} \right\}$$

0 otherwise

5U. Compute the Fourier transform of each of the following signals:

a) $x[n] = u[n+2] - u[n-5]$

b) $x[n] = (1/5)^{|n|} u[-n-6]$

c) $x[n] = (1/3)^{|n|} \cos\{(\pi/6)(n+3)\}$

d) $x[n] = \sin(n\pi/3) + \cos(n)$

Solution:

6S. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.

a)

$$X(e^{jw}) = 1 \quad \text{for } \frac{\pi}{4} \leq |w| \leq \frac{3\pi}{4}$$

$$0 \quad \text{for } \frac{3\pi}{4} \leq |w| \leq \pi \quad \text{and} \quad 0 \leq |w| \leq \frac{\pi}{4}$$

b) $X(e^{jw}) = e^{-jw/2} \quad \text{for } -\pi \leq w \leq \pi$

c) $X(e^{jw}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(w - \frac{\pi}{2}k)$

d) $X(e^{jw}) = \frac{1 - \frac{1}{3}e^{-jw}}{1 - \frac{2}{3}e^{-jw} + \frac{1}{9}e^{-2jw}}$

Solution:

a)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw = \frac{1}{2\pi} \int_{\pi/4}^{3\pi/4} e^{jwn} dw + \frac{1}{2\pi} \int_{-3\pi/4}^{-\pi/4} e^{jwn} dw$$

$$x[n] = \frac{1}{2\pi(jn)} \{e^{jn(\frac{3\pi}{4})} - e^{jn(\frac{\pi}{4})}\} + \frac{1}{2\pi(jn)} \{e^{jn(\frac{-\pi}{4})} - e^{jn(\frac{-3\pi}{4})}\}$$

$$x[n] = \frac{1}{2\pi(jn)} \{2j \sin(n\frac{3\pi}{4}) - 2j \sin(n\frac{\pi}{4})\} = \frac{1}{\pi n} \{\sin(n\frac{3\pi}{4}) - \sin(n\frac{\pi}{4})\}$$

b)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jw/2} e^{jwn} dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(n-\frac{1}{2})w} dw$$

$$x[n] = \frac{1}{2\pi j(n-\frac{1}{2})} (e^{j(n-\frac{1}{2})\pi} - e^{-j(n-\frac{1}{2})\pi}) = \frac{2j}{2\pi j(n-\frac{1}{2})} \sin(n\pi - \frac{\pi}{2})$$

$$x[n] = \frac{1}{\pi(n-\frac{1}{2})} \sin(n\pi - \frac{\pi}{2})$$

c) $X(e^{jw}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(w - \frac{\pi}{2}k)$ is periodic with fundamental period of 4 therefore

fundamental frequency $w_0 = \frac{\pi}{2}$, also $a_k = \frac{1}{2\pi} (-1)^k$. Using these facts we could use the Fourier series synthesis equation to find $x[n]$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n} = \sum_{k=0}^3 a_k e^{jk(\frac{2\pi}{N})n} = \frac{1}{2\pi} - \frac{1}{2\pi} e^{j\pi n/2} + \frac{1}{2\pi} e^{j\pi n} - \frac{1}{2\pi} e^{j3\pi n/2}$$

d) Rewrite the function by doing a long division

$$X(e^{jw}) = \frac{1 - \frac{1}{3}e^{-jw}}{1 - \frac{2}{3}e^{-jw} + \frac{1}{9}e^{-2jw}} = \frac{1 - \frac{1}{3}e^{-jw}}{\left(1 - \frac{1}{3}e^{-jw}\right)^2} = \frac{1}{1 - \frac{1}{3}e^{-jw}}$$

$$\text{Given} \Rightarrow \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{where } a < 1$$

$$X(e^{jw}) = 1 + \frac{1}{3}e^{-jw} + \frac{1}{3^2}e^{-j2w} + \frac{1}{3^3}e^{-j3w} + \frac{1}{3^4}e^{-j4w} + \frac{1}{3^5}e^{-j5w} + \dots$$

from F.T. table

$$\text{From table } e^{-jwn_0} \xrightarrow{\text{I.F.T.}} \delta(n - n_0)$$

$$X[n] = \delta(n) + \frac{1}{3}\delta(n-1) + \frac{1}{3^2}\delta(n-2) + \frac{1}{3^3}\delta(n-3) + \frac{1}{3^4}\delta(n-4) + \frac{1}{3^5}\delta(n-5) + \dots$$

6U. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.

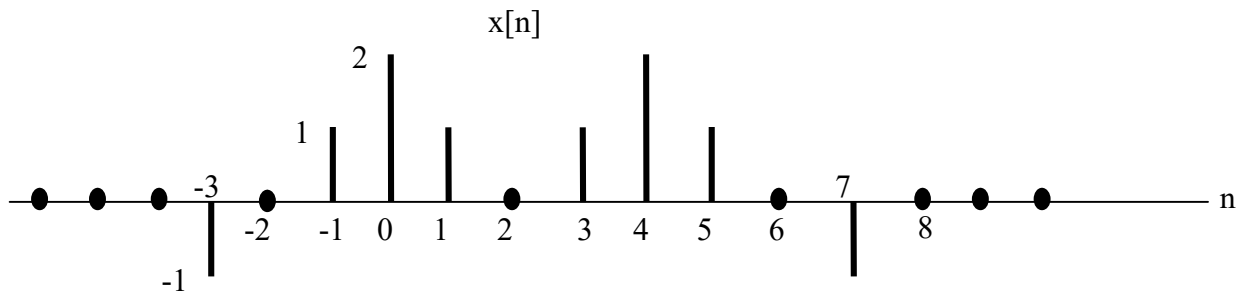
$$\text{a) } X(e^{jw}) = \begin{cases} e^{-jw/3} & \text{for } -2\pi/3 \leq w \leq 2\pi/3 \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{b) } X(e^{jw}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(w - \frac{\pi k}{3})$$

$$\text{c) } X(e^{jw}) = \frac{1 + \frac{1}{5}e^{-jw}}{-1 - \frac{2}{5}e^{-jw} - \frac{1}{25}e^{-2jw}}$$

Solution:

7S. Let $X(e^{jw})$ denote the Fourier transform of the signal $x[n]$ depicted in the following figure:



Perform the following calculations without explicitly evaluating $X(e^{jw})$:

- Evaluate $X(e^{j0})$
- Find the phase of $X(e^{jw})$
- Evaluate $\int_{-\pi}^{\pi} X(e^{jw}) dw$
- Find $X(e^{j\pi})$
- Determine and sketch the signal whose Fourier transform is $\text{Real}\{X(e^{jw})\}$
- Evaluate $\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw$
- Evaluate $\int_{-\pi}^{\pi} \left| \frac{dX(e^{jw})}{dw} \right|^2 dw$

Solution:

a) write the Fourier Transform Equation

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n] e^0 = -1 + 0 + 1 + 2 + 1 + 0 + 1 + 2 + 1 + 0 - 1 = 6$$

- b) Note that the function $y[n]=x[n+2]$ is an even signal, therefore $Y(e^{jw})$ is real and even. This implies that the phase of $Y(e^{jw})=0$.
Using the time shifting property of the Fourier transform we have $Y(e^{jw})=e^{j2w}X(e^{jw})$ therefore phase of $Y(e^{jw})$ is $-2w$.

c)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Inverse Fourier Transform}$$

$$2\pi x[0] = \int_{2\pi} X(e^{j\omega}) d\omega$$

$$\text{We have } x[0] = 2 \Rightarrow \int_{2\pi} X(e^{j\omega}) d\omega = 4\pi$$

d)

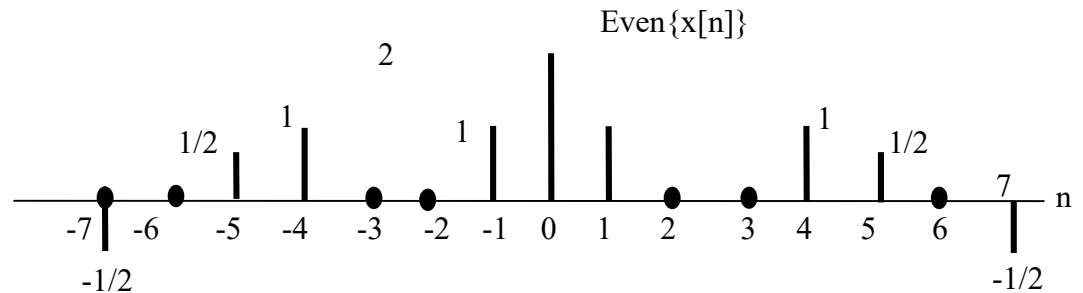
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\pi n} = \sum_{n=-\infty}^{+\infty} x[n] (-1)^n = +1 + 0 - 1 + 2 - 1 + 0 - 1 + 2 - 1 + 0 + 1 = 2$$

e) From property table we have the following

$$\text{even}\{x[n]\} \xrightarrow{FT} \text{Re}\{X(e^{j\omega})\}$$

Therefore, the desired signal is $\text{even}\{x[n]\} = (x[n] + x[-n]) / 2$ shown below:



f) From Parseval theorem we have

$$\frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = 14 \Rightarrow \int_{2\pi} |X(e^{j\omega})|^2 d\omega = 28\pi$$

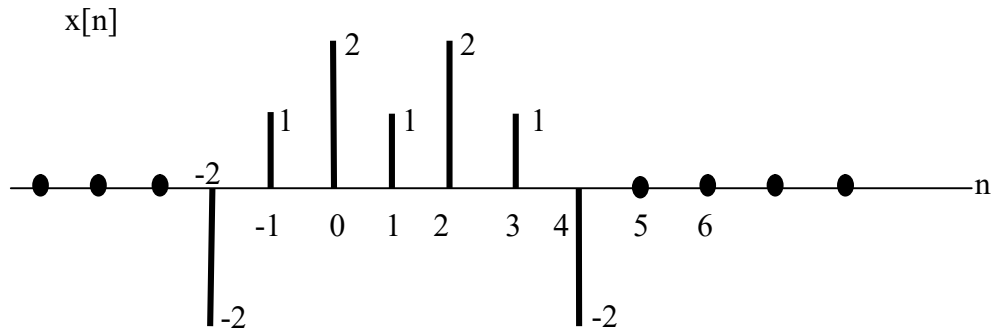
g) Using the differentiation in frequency property of Fourier transform

$$nx[n] \xrightarrow{FT} j \frac{dX(e^{j\omega})}{d\omega}$$

Use Parseval's theorem

$$\frac{1}{2\pi} \int_{2\pi} \left| j \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = \sum_{n=-\infty}^{+\infty} |nx[n]|^2 = 158 \Rightarrow \int_{2\pi} \left| j \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 316\pi$$

7U. Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted in the following figure:



Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

- Evaluate $X(e^{j0})$
- Find the phase of $X(e^{j\omega})$
- Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$
- Find $X(e^{j\pi})$
- Determine and sketch the signal whose Fourier transform is Pure Imaginary $\{X(e^{j\omega})\}$
- Evaluate $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$
- Evaluate $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega$

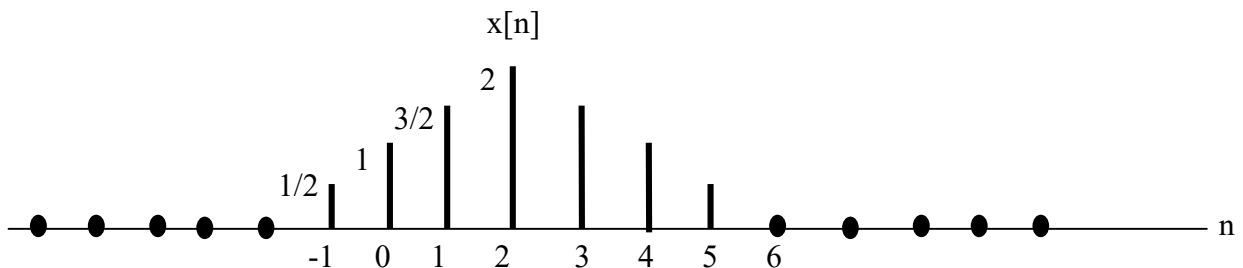
Solution:

8S. Six conditions are listed below:

- $\text{Real}\{X(e^{j\omega})\} = 0$
- $\text{Imaginary}\{X(e^{j\omega})\} = 0$
- There exist a real p such that $e^{jp\omega} X(e^{j\omega})$ is real.
- $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$ is periodic
- Periodic $X(e^{j\omega})$
- $X(e^{j0}) = 0$

Which, if any, of the following signals have Fourier transforms that satisfy each of the above conditions:

a) $x[n]$ Shown in the following diagram



- $x[n] = (1/2)^{|n|}$
- $x[n] = \delta[n - 1] + \delta[n + 3]$

Solution:

Facts:

- 1) For $\text{Re}\{X(e^{j\omega})\}$ to be zero, the signal must be real and odd.
- 2) For $\text{Im}\{X(e^{j\omega})\}$ to be zero, the signal must be real and even.
- 3) Assume $Y(e^{j\omega}) = e^{ja\omega}X(e^{j\omega})$. Using the time shifting property of the Fourier transform we have $y[n]=x[n+a]$. If $Y(e^{j\omega})$ is real, then $y[n]$ is real and even (assuming that $x[n]$ is real). Therefore $x[n]$ has to be symmetric about a.
- 4) Since $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0]$. The given condition is satisfied only if $x[0]=0$.
- 5) $X(e^{j\omega})$ is always periodic with period 2π . Therefore, all signals satisfy this condition.
- 6) Since $X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n]$, the given condition is satisfied only if samples of the signal add up to zero.

	Cond. #1	Cond. #2	Cond. #3	Cond. #4	Cond. #5	Cond. #6
Part a			Yes		Yes	
Part b		Yes	Yes		Yes	
Part c					Yes	

8U. Six conditions are listed below:

- 1) $\text{Real}\{X(e^{j\omega})\} = 0$
- 2) $\text{Imaginary}\{X(e^{j\omega})\} = 0$
- 3) There exist a real a such that $e^{ja\omega} X(e^{j\omega})$ is real.
- 4) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$ is periodic
- 5) Periodic $X(e^{j\omega})$
- 6) $X(e^{j0}) = 0$

Which, if any, of the following signals have Fourier transforms that satisfy each of the above conditions:

- a) $x[n] = (1/3)^{|n|}$
- b) $x[n] = \delta[n + 1] + \delta[n - 3]$

Solution:

9S. Using $X(e^{j\omega})$ as the Fourier transform of $x[n]$, express the Fourier transform of the following signal in terms of $X(e^{j\omega})$.

$$y[n] = (n - 1)^2 x[n]$$

Solution:

$$\text{Differentiation} \Rightarrow nx[n] \xrightarrow{FT} j \frac{dX(e^{j\omega})}{d\omega}$$

$$\text{Differentiation} \Rightarrow n^2 x[n] \xrightarrow{FT} -\frac{d^2 X(e^{j\omega})}{d\omega^2}$$

$$\text{Therefore} \Rightarrow y[n] = [n^2 - 2n + 1]x[n] \Rightarrow Y(e^{j\omega}) = -\frac{d^2 X(e^{j\omega})}{d\omega^2} - 2j \frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

9U. Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transform of the following signal in terms of $X(e^{j\omega})$:

$$y[n] = (n^2 + 3n + 5)x[n]$$

Solution:

10S. Use the inverse Fourier transform integral to determine the inverse Fourier transform of:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \left\{ 2\pi\delta(\omega - 2\pi k) + \pi\delta\left(\omega - \frac{\pi}{2} - 2\pi k\right) + \pi\delta\left(\omega + \frac{\pi}{2} - 2\pi k\right) \right\}$$

Solution:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Synthesis equation}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \left\{ 2\pi\delta(\omega - 2\pi k) + \pi\delta\left(\omega - \frac{\pi}{2} - 2\pi k\right) + \pi\delta\left(\omega + \frac{\pi}{2} - 2\pi k\right) \right\} e^{j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ 2\pi\delta(\omega) + \pi\delta\left(\omega - \frac{\pi}{2}\right) + \pi\delta\left(\omega + \frac{\pi}{2}\right) \right\} e^{j\omega n} d\omega$$

$$x[n] = e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} = 1 + \cos(\pi n / 2)$$

10U. Express the Fourier transform of the following signal in terms of $X(e^{j\omega})$:

$$y[n] = \frac{x^*[-n] + x[n]}{2}$$

Solution: